

Some New bounds on the Delta and Reverse Zagreb Indices

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Abstract: In this paper relationships between the first Zagreb index and delta (reverse) first Zagreb index are derived. Also some new bounds for these topological indices are obtained.

Keywords: Topological indices, degree-based invariants, Zagreb indices

1 Introduction

Let $G = (V, E)$, $V = \{v_1, v_2, \dots, v_n\}$, be a simple graph with n vertices, m edges with vertex-degree sequence $\Delta = d_1 \geq d_2 \geq \dots \geq d_n = \delta$, $d_i = d(v_i)$. If vertices v_i and v_j are adjacent in G , we write $i \sim j$, otherwise we write $i \not\sim j$. As usual, $\overline{G}$ denotes complement of the graph G .

Vertex-degree-based topological indices can be represented in the following form [11]

$$TI(G) = \sum_{i \sim j} F(d_i, d_j), \quad (1.1)$$

where $F(x, y)$ is an appropriately chosen function with the property $F(x, y) = F(y, x)$. Let f be a function of the set $D = \{d_1, d_2, \dots, d_n\}$ into the set of non-negative real numbers. Then a class of vertex-degree-based topological indices, defined by (1.1), can be extended in the following way

$$TI_f(G) = \sum_{i \sim j} F(f(d_i), f(d_j)). \quad (1.2)$$

Manuscript received May 13, 2022.; accepted October 21, 2022.

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In this paper we will consider the case when $F(x, y)$ is an additive function, $F(x, y) = x + y$ (see, for example, [5]). In that case (1.2) becomes

$$TI_f(G) = \sum_{i \sim j} (f(d_i) + f(d_j)) = \sum_{i=1}^n d_i f(d_i). \quad (1.3)$$

Here we mention some particular cases. When $f(x) = x$, we get

$$M_1(G) = \sum_{i \sim j} (d_i + d_j) = \sum_{i=1}^n d_i^2,$$

the first Zareb index [12]. It is one of the most popular and most extensively studied graph-based molecular structure descriptors. More on its mathematical properties and chemical applications can be found in [2, 13, 14, 21] and in the references cited therein. Additive form of this index was introduced in [21].

Various modifications of the first Zagreb index have been proposed. For $f(x) = x - \delta + 1$, $x \in D$, from (1.3) we have

$$\delta M_1^\beta(G) = \sum_{i \sim j} (s_i + s_j) = \sum_{i=1}^n d_i s_i, \quad (1.4)$$

where

$$s_i = d_i - \delta + 1, \quad i = 1, 2, \dots, n, \quad (1.5)$$

which is known as delta first Zagreb beta index [17]. In the same paper, delta first Zagreb alpha index, $\delta M_1^\alpha(G)$, was encountered as well. It was conceived as

$$\delta M_1^\alpha(G) = \sum_{i=1}^n s_i^2. \quad (1.6)$$

For $f(x) = \Delta - x + 1$, $x \in D$, based on (1.3) we have

$$RM_1^\beta(G) = \sum_{i \sim j} (c_i + c_j) = \sum_{i=1}^n d_i c_i, \quad (1.7)$$

where

$$c_i = \Delta - d_i + 1, \quad i = 1, 2, \dots, n, \quad (1.8)$$

is the so called reverse first Zagreb beta index [7] (see also [8, 9]). Also, in these papers reverse first Zagreb alpha index, $RM_1^\alpha(G)$, was defined as

$$RM_1^\alpha(G) = \sum_{i=1}^n c_i^2. \quad (1.9)$$

A concept of coindex of a graph was introduced in [6] (see also [1]). In this case the sum runs over the edges of the complement of G . In a view of (1.3) the corresponding coindex of G can be defined as

$$\overline{TI}_f(G) = \sum_{i \sim j} (f(d_i) + f(d_j)) = \sum_{i=1}^n (n-1-d_i) f(d_i).$$

For $f(x) = x$ the first Zagreb coindex, $\overline{M}_1(G)$, is obtained (see [1, 6])

$$\overline{M}_1(G) = \sum_{i=1}^n (n-1-d_i) d_i. \quad (1.10)$$

By analogy with the first Zagreb coindex, delta first Zagreb beta coindex, $\overline{\delta M}_1^\beta(G)$, and reverse first Zagreb beta coindex, $\overline{RM}_1^\beta(G)$, are defined as

$$\overline{\delta M}_1^\beta(G) = \sum_{i \sim j} (s_i + s_j) = \sum_{i=1}^n (n-1-d_i) s_i \quad (1.11)$$

and

$$\overline{RM}_1^\beta(G) = \sum_{i \sim j} (c_i + c_j) = \sum_{i=1}^n (n-1-d_i) c_i. \quad (1.12)$$

In this paper we investigate a relationships between the first Zagreb index and delta (reverse) first Zagreb index. Some new bounds for these topological indices are obtained, as well.

2 Preliminaries

At the beginning we recall some analytical inequalities for real number sequences which will be used in the rest of the paper.

In [15] (see also [18]) the following result was proven.

Lemma 2.1. [15] *Let $p = (p_i)$, $i = 1, 2, \dots, n$, be positive real number sequence, $a = (a_i)$ and $b = (b_i)$, $i = 1, 2, \dots, n$, be nonnegative real number sequences. If $a = (a_i)$ and $b = (b_i)$ are of similar monotonicity, then*

$$\sum_{i=1}^n p_i \sum_{i=1}^n p_i a_i b_i \geq \sum_{i=1}^n p_i a_i \sum_{i=1}^n p_i b_i. \quad (2.1)$$

If $a = (a_i)$ and $b = (b_i)$ are of opposite monotonicity, then the opposite inequality holds. Equality holds if and only if $a_1 = a_2 = \dots = a_n$ or $b_1 = b_2 = \dots = b_n$.

The inequality (2.1) is known in the literature as Chebyshev's inequality. More on this inequality can be found, for example, in [19].

Lemma 2.2. [20] *Let $p = (p_i)$, $i = 1, 2, \dots, n$, be a sequence of nonnegative real numbers and $a = (a_i)$, $i = 1, 2, \dots, n$, sequence of positive real numbers. Then, for any r , $r \leq 0$ or $r \geq 1$, holds*

$$\left(\sum_{i=1}^n p_i \right)^{r-1} \sum_{i=1}^n p_i a_i^r \geq \left(\sum_{i=1}^n p_i a_i \right)^r. \quad (2.2)$$

When $0 \leq r \leq 1$ the opposite inequality is valid. Equality holds if and only if either $r = 0$, or $r = 1$, or $a_1 = a_2 = \dots = a_n$, or $p_1 = p_2 = \dots = p_t = 0$ and $a_{t+1} = \dots = a_n$, or $p_{t+1} = \dots = p_n = 0$ and $a_1 = a_2 = \dots = a_t$, for some t , $1 \leq t \leq n-1$.

The inequality (2.2) is known as Jensen's inequality. Original form of this inequality was proven in [16].

3 A relationship between First Zagreb indices/coindices of graphs

In this section we consider relationships between the first Zagreb indices (coindices) introduced in Section 1.

Lemma 3.1. *For every graph G with the property $\delta = 1$ holds*

$$M_1(G) = \delta M_1^\alpha(G) = \delta M_1^\beta(G). \quad (3.1)$$

Proof. If G is a graph such that $\delta = 1$, then for any $i = 1, 2, \dots, n$ we have that $s_i = d_i$, from which directly follows (3.1). $\square$

In general, equalities in (3.1) do not hold. In another cases we have the following result.

Lemma 3.2. *Let G be a graph with $n \geq 2$ vertices and m edges. Then*

$$\delta M_1^\alpha(G) = M_1(G) - 4m(\delta - 1) + n(\delta - 1)^2, \quad (3.2)$$

$$\delta M_1^\beta(G) = M_1(G) - 2m(\delta - 1) \quad (3.3)$$

and

$$\delta M_1^\beta(G) = \delta M_1^\alpha(G) + 2m(\delta - 1) - n(\delta - 1)^2. \quad (3.4)$$

Proof. According to (1.6) and (1.4) we have

$$\begin{aligned} \delta M_1^\alpha(G) &= \sum_{i=1}^n s_i^2 = \sum_{i=1}^n (d_i - \delta + 1)^2 = \sum_{i=1}^n (d_i^2 - 2d_i(\delta - 1) + (\delta - 1)^2) \\ &= M_1(G) - 4m(\delta - 1) + n(\delta - 1)^2, \end{aligned}$$

$$\delta M_1^\beta(G) = \sum_{i \sim j} (s_i + s_j) = \sum_{i=1}^n d_i s_i = \sum_{i=1}^n d_i (d_i - \delta + 1) = M_1(G) - 2m(\delta - 1),$$

which is stated by (3.2) and (3.3), and (3.4) follows directly from (3.2) and (3.3). $\square$

Corollary 3.1. *Let G be a graph with $n \geq 2$ vertices and m edges. Then*

$$\delta M_1^\alpha(G) \leq 2m(\Delta - \delta + 2) + n((\delta - 1)^2 - \Delta\delta)$$

and

$$\delta M_1^\beta(G) \leq 2m(\Delta + 1) - n\Delta\delta.$$

Equalities hold if and only if $d_i \in \{\Delta, \delta\}$ for every $i = 1, 2, \dots, n$.

Proof. In [3] the following inequality was proven

$$M_1(G) \leq 2m(\Delta + \delta) - n\Delta\delta, \quad (3.5)$$

with equality if and only if $d_i \in \{\Delta, \delta\}$ for every $i = 1, 2, \dots, n$. From the above and (3.2) and (3.3) we get the desired results. $\square$

Corollary 3.2. *Let G be a graph with $n \geq 2$ vertices, m edges with the property $\delta = 1$. Then*

$$\delta M_1^\alpha(G) = \delta M_1^\beta(G) \leq 2m(\Delta + 1) - n\Delta.$$

Equality holds if and only if $d_i \in \{\Delta, 1\}$ for every $i = 1, 2, \dots, n$.

The proof of the next lemma is analogous to that of Lemma 3.2, hence omitted.

Lemma 3.3. *Let G be a graph with $n \geq 2$ vertices and m edges. Then*

$$RM_1^\alpha(G) = M_1(G) - 4m(\Delta + 1) + n(\Delta + 1)^2, \quad (3.6)$$

$$RM_1^\beta(G) = 2m(\Delta + 1) - M_1(G) \quad (3.7)$$

and

$$RM_1^\alpha(G) + RM_1^\beta(G) = n(\Delta + 1)^2 - 2m(\Delta + 1). \quad (3.8)$$

Lemma 3.4. *Let G be a graph with $n \geq 2$ vertices and m edges. Then*

$$\delta M_1^\alpha(G) - RM_1^\alpha(G) = (\Delta - \delta + 2)(4m - n(\Delta + \delta)),$$

$$\delta M_1^\alpha(G) + RM_1^\beta(G) = 2m(\Delta - 2\delta + 3) + n(\delta - 1)^2,$$

$$\delta M_1^\beta(G) - RM_1^\alpha(G) = 2m(2\Delta - \delta + 3) - n(\Delta + 1)^2,$$

$$\delta M_1^\beta(G) + RM_1^\beta(G) = 2m(\Delta - \delta + 2).$$

In the following lemma we determine a relationship between sequences $s = (s_i)$ and $c = (c_i)$, $i = 1, 2, \dots, n$, defined by (1.5) and (1.8), respectively.

Lemma 3.5. *For every $i = 1, 2, \dots, n$ holds*

$$c_i = c_i(G) = s_i(\overline{G}) = \overline{s}_i. \quad (3.9)$$

Proof. For every $i = 1, 2, \dots, n$ the following is valid

$$\begin{aligned} c_i &= c_i(G) = \Delta - d_i + 1 = n - 1 - d_i - (n - 1 - \Delta) + 1 \\ &= d_i(\overline{G}) - \delta(\overline{G}) + 1 = s_i(\overline{G}) = \bar{s}_i, \end{aligned}$$

which is desired result. $\square$

The equality (3.9) enables us to obtain Nordhaus–Gaddum type identities (see [22]) for topological indices $RM_1^\alpha(G)$ and $RM_1^\beta(G)$, i.e. $\delta M_1^\alpha(G)$ and $\delta M_1^\beta(G)$.

Lemma 3.6. *Let G be a graph with $n \geq 2$ vertices and m edges. Then*

$$\begin{aligned} RM_1^\alpha(G) + RM_1^\alpha(\overline{G}) &= \delta M_1^\alpha(G) + \delta M_1^\alpha(\overline{G}) \\ &= 2M_1(G) - 4m(\Delta + \delta) + n((\Delta + 1)^2 + (\delta - 1)^2) \end{aligned} \quad (3.10)$$

and

$$RM_1^\beta(G) + RM_1^\beta(\overline{G}) = \delta M_1^\beta(G) + \delta M_1^\beta(\overline{G}) = 2m(\Delta - \delta + 2). \quad (3.11)$$

Proof. Based on (3.9) we have

$$\delta M_1^\alpha(G) = \sum_{i=1}^n s_i^2 = \sum_{i=1}^n \bar{c}_i^2 = RM_1^\alpha(\overline{G})$$

and

$$\delta M_1^\beta(G) = \sum_{i \sim j} (s_i + s_j) = \sum_{i \sim j} (\bar{c}_i + \bar{c}_j) = RM_1^\beta(\overline{G}),$$

from which it follows

$$RM_1^\alpha(G) + RM_1^\alpha(\overline{G}) = \delta M_1^\alpha(G) + \delta M_1^\alpha(\overline{G}) = RM_1^\alpha(G) + \delta M_1^\alpha(G)$$

and

$$RM_1^\beta(G) + RM_1^\beta(\overline{G}) = \delta M_1^\beta(G) + \delta M_1^\beta(\overline{G}) = RM_1^\beta(G) + \delta M_1^\beta(G).$$

From the above and (3.2), (3.3), (3.6) and (3.7) we get the desired results. $\square$

Corollary 3.3. *Let G be a graph with $n \geq 2$ vertices and m edges. Then*

$$RM_1^\alpha(G) + RM_1^\alpha(\overline{G}) = \delta M_1^\alpha(G) + \delta M_1^\alpha(\overline{G}) \leq n((\Delta + 1)^2 + (\delta - 1)^2 - 2\Delta\delta).$$

Equality holds if and only if $d_i \in \{\Delta, \delta\}$ for every $i = 1, 2, \dots, n$.

Proof. The desired result immediately follows from (3.10) and (3.5). $\square$

The following relation between the first Zagreb index and coindex was determined in [4].

Lemma 3.7. [4] *Let G be a graph with $n \geq 2$ vertices and m edges. Then*

$$\overline{M}_1(G) + M_1(G) = 2m(n-1).$$

Similarly, for $\delta M_1^\beta(G)$ and $RM_1^\beta(G)$ the following is valid.

Lemma 3.8. *Let G be a graph with $n \geq 2$ vertices and m edges. Then*

$$\overline{\delta M}_1^\beta(G) + \delta M_1^\beta(G) = (n-1)(2m - n(\delta-1))$$

and

$$\overline{RM}_1^\beta(G) + RM_1^\beta(G) = (n-1)(n(\Delta+1) - 2m).$$

4 Some new bounds on the reverse first Zagreb indices

In the next theorem we determine an upper bound for $RM_1^\beta(G)$ in terms of graph structural parameters n , m , Δ and δ .

Theorem 4.1. *Let G be a graph with $n \geq 3$ vertices and m edges. Then*

$$RM_1^\beta(G) \leq \Delta + \delta(\Delta - \delta + 1) + \frac{(2m - \Delta - \delta)(n(\Delta + 1) - 2m - (\Delta - \delta + 2))}{n-2}. \quad (4.1)$$

Equality holds if and only if $d_2 = d_3 = \dots = d_{n-1}$.

Proof. If $a = (a_i)$ and $b = (b_i)$, $i = 2, 3, \dots, n-1$, are of opposite monotonicity, then (2.1) can be considered in the following form

$$\sum_{i=2}^{n-1} p_i \sum_{i=2}^{n-1} p_i a_i b_i \leq \sum_{i=2}^{n-1} p_i a_i \sum_{i=2}^{n-1} p_i b_i.$$

Since (d_i) and (c_i) , $i = 2, 3, \dots, n-1$, are of opposite monotonicity, for $p_i = 1$, $a_i = d_i$, $b_i = c_i$, $i = 2, 3, \dots, n-1$, the above inequality becomes

$$\sum_{i=2}^{n-1} 1 \sum_{i=2}^{n-1} d_i c_i \leq \sum_{i=2}^{n-1} d_i \sum_{i=2}^{n-1} c_i,$$

i.e.

$$(n-2) \left(\sum_{i=1}^n d_i c_i - d_1 c_1 - d_n c_n \right) \leq \left(\sum_{i=1}^n d_i - d_1 - d_n \right) \left(\sum_{i=1}^n c_i - c_1 - c_n \right). \quad (4.2)$$

Since

$$\sum_{i=1}^n d_i = 2m, \quad d_1 = \Delta, \quad d_n = \delta,$$

and

$$\sum_{i=1}^n c_i = \sum_{i=1}^n (\Delta - d_i + 1) = n(\Delta + 1) - 2m, \quad c_1 = 1, \quad c_n = \Delta - \delta + 1, \quad (4.3)$$

from (4.2) it follows

$$(n-2)(RM_1^\beta(G) - \Delta - \delta(\Delta - \delta + 1)) \leq (2m - \Delta - \delta)(n(\Delta + 1) - 2m - 1 - (\Delta - \delta + 1)),$$

from which we get (4.1).

Equality in (4.2) holds if and only if $d_2 = d_3 = \dots = d_{n-1}$ or $c_2 = c_3 = \dots = c_{n-1}$, therefore equality in (4.1) holds if and only if $d_2 = d_3 = \dots = d_{n-1}$. $\square$

The proof of the next theorem is analogous to that of Theorem 4.1, hence omitted.

Theorem 4.2. *Let G be a graph with $n \geq 3$ vertices and m edges. Then*

$$RM_1^\alpha(G) \geq 1 + (\Delta - \delta + 1)^2 + \frac{(n(\Delta + 1) - 2m - (\Delta - \delta + 2))^2}{n - 2}.$$

Equality holds if and only if $d_2 = d_3 = \dots = d_{n-1}$.

The forgotten topological index, $F(G)$, encountered in [10] is defined as

$$F(G) = \sum_{i=1}^n d_i^3.$$

In the next theorem we determine an upper bound for $RM_1^\beta(G)$ in terms of topological indices $M_1(G)$ and $F(G)$ and parameters n , m and Δ .

Theorem 4.3. *Let G be a simple graph with n vertices and m edges. Then*

$$RM_1^\beta(G) \leq \min \left\{ \sqrt{(n(\Delta + 1) - 2m)((\Delta + 1)M_1(G) - F(G))}, \right. \\ \left. \sqrt{2m(F(G) - 2(\Delta + 1)M_1(G) + 2m(\Delta + 1)^2)} \right\}. \quad (4.4)$$

Equality holds if and only if G is a regular graph.

Proof. For $r = 2$, $p_i = c_i$, $a_i = d_i$, $i = 1, 2, \dots, n$, the inequality (2.2) transforms into

$$\sum_{i=1}^n c_i \sum_{i=1}^n c_i d_i^2 \geq \left(\sum_{i=1}^n d_i c_i \right)^2. \quad (4.5)$$

Since

$$\sum_{i=1}^n c_i d_i^2 = \sum_{i=1}^n (\Delta - d_i + 1) d_i^2 = (\Delta + 1)M_1(G) - F(G),$$

from (4.5) it follows

$$RM_1^\beta(G) \leq \sqrt{(n(\Delta + 1) - 2m)((\Delta + 1)M_1(G) - F(G))}. \quad (4.6)$$

Equality in (4.5) is attained if and only if $d_1 = d_2 = \dots = d_n$, which means that equality in (4.6) holds if and only if G is a regular graph.

Similarly, for $r = 2$, $p_i = d_i$, $a_i = c_i$, $i = 1, 2, \dots, n$, the inequality (2.2) becomes

$$\sum_{i=1}^n d_i \sum_{i=1}^n d_i (\Delta - d_i + 1)^2 \geq \left(\sum_{i=1}^n d_i c_i \right)^2. \quad (4.7)$$

Since

$$\sum_{i=1}^n d_i (\Delta - d_i + 1)^2 = F(G) - 2(\Delta + 1)M_1(G) + 2m(\Delta + 1)^2,$$

and from (4.7) it follows

$$RM_1^\beta(G) \leq \sqrt{2m(F(G) - 2(\Delta + 1)M_1(G) + 2m(\Delta + 1)^2)}. \quad (4.8)$$

Equality in (4.7) is attained if and only if $c_1 = c_2 = \dots = c_n$, which means that equality in (4.8) holds if and only if G is a regular graph.

Finally, from (4.6) and (4.8) we arrive at (4.4). $\square$

In a similar way as in case of Theorem 4.3, we obtain the next result.

Theorem 4.4. *Let G be a graph with $n \geq 3$ vertices and m edges. Then*

$$\begin{aligned} RM_1^\alpha(G) &\leq 1 + (\Delta - \delta + 1)^2 \\ &+ \sqrt{(n(\Delta + 1) - 2m - (\Delta - \delta + 2))} \\ &\times \sqrt{(n(\Delta + 1)^3 - 6m(\Delta + 1)^2 + 3(\Delta + 1)M_1(G) - F(G) - 1 - (\Delta - \delta + 1)^3)}. \end{aligned}$$

Equality holds if and only if $d_2 = d_3 = \dots = d_{n-1}$.

The next two theorems directly follow from definitions of topological indices $RM_1^\beta(G)$ and $RM_1^\alpha(G)$.

Theorem 4.5. *Let G be a graph with n vertices and m edges. Then*

$$\max\{2m, \delta(n(\Delta + 1) - 2m)\} \leq RM_1^\beta(G) \leq \max\{2m(\Delta - \delta + 1), \Delta(n(\Delta + 1) - 2m)\}.$$

Equalities hold if and only if G is a regular graph.

Theorem 4.6. *Let G be a graph with n vertices and m edges. Then*

$$n(\Delta + 1) - 2m \leq RM_1^\alpha(G) \leq (\Delta - \delta + 1)(n(\Delta + 1) - 2m).$$

Equalities hold if and only if G is a regular graph.

Acknowledgement

Research supported by the Serbian Ministry of Education, Science and Technological Development, grant No. 451-03-68/2022-14/ 200102.

References

- [1] A. R. ASHRAFI, T. DOŠLIĆ, A. HAMZEH, *The Zagreb coindex of graph operations*, Discr. Appl. Math. 158 (2010) 1571–1578.
- [2] B. BOROVIĆANIN, K. C. DAS, B. FURTULA, I. GUTMAN, *Bounds for the Zagreb indices*, MATCH Commun. Math. Comput. Chem. 78(1) (2017) 17–100.
- [3] K. C. DAS, *Maximizing the sum of squares of the degrees of a graph*, Discrete Math. 285(1–3) (2004) 57–66.
- [4] K. C. DAS, I. GUTMAN, *Some properties of the second Zagreb index*, MATCH Commun. Math. Comput. Chem. 52(1) (2004) 103–112.
- [5] T. DOŠLIĆ, T. RETI, D. VUKIČEVIĆ, *On the vertex degree indices of connected graphs*, Chem. Phys. Lett. 512 (2011) 283–286.
- [6] T. DOŠLIĆ, *Vertex-weighted Wiener polynomials for composite graphs*, Ars Math. Contemp. 1 (2008) 66–80.
- [7] S. EDIZ, M. CANGCAN, *Reverse Zagreb indices of Cartesian product of graphs*, Int. J. Math. Comput. Sci. 11 (2016) 51–58.
- [8] S. EDIZ, *Maximal chemical trees of the second reverse Zagreb index*, Pacific J. Appl. Math. 7 (2015) 291–295.
- [9] S. EDIZ, *Maximal graphs of the first reverse Zagreb beta index*, TWMS J. Appl. Eng. Math. 8(1a) (2018) 306–310.
- [10] B. FURTULA, I. GUTMAN, *A forgotten topological index*, J. Math. Chem. 53 (2015) 1184–1190.
- [11] I. GUTMAN, *Degree-based topological indices*, Croat. Chem. Acta 86 (2013) 351–361.
- [12] I. GUTMAN, N. TRINAJSTIĆ, *Graph theory and molecular orbitals. Total π -electron energy of alternant hydrocarbons*, Chem. Phys. Lett. 17 (1972) 535–538.
- [13] I. GUTMAN, K. CH. DAS, *The first Zagreb index 30 years later*, MATCH Commun. Math. Comput. Chem. 50 (2004) 83–92.
- [14] I. GUTMAN, E. MILOVANOVIĆ, I. MILOVANOVIĆ, *Beyond the Zagreb indices*, AKCE Int. J. Graphs Comb. 17(1) (2020) 74–85.
- [15] T. HAYASHI, *On some inequalities*, Rend. Circ. Mat. Palermo 44 (1920) 336–340.
- [16] J. L. W. V. JENSEN, *Sur les fonctions convexes et les inegalites entre les valeurs moyennes*, Acta Math. 30 (1906) 175–193.
- [17] V. R. KULLI, *δ -Sombor index and its exponential for certain nanotubes*, Annals Pure Appl. Math. 23 (2021) 37–42.
- [18] D. S. MITRINOVIĆ, P. M. VASIĆ, *Analytic inequalities*, Springer Verlag, Berlin–Heidelberg–New York, 1970.

- [19] D. S. MITRINOVIĆ, P. M. VASIĆ, *History, variations and generalisations of the Čebyšev inequality and the question of some priorities*, Univ. Beograd Publ. Elektrotehn. Fak. Ser. Mat. Fiz. 461-497 (1974) 1–30.
- [20] D. S. MITRINOVIĆ, J. E. PEČARIĆ, A. M. FINK, *Classical and new inequalities in analysis*, Kluwer Academic Publishers, Dordrecht, 1993.
- [21] S. NIKOLIĆ, G. KOVAČEVIĆ, A. MILIČEVIĆ, N. TRINAJSTIĆ, *The Zagreb indices 30 years after*, Croat. Chem. Acta 76 (2003) 113–124.
- [22] E. A. NORDHAUS, J. W. GADDUM, *On complementary graphs*, Amer. Math. Monthly 63(3) (1956) 175–177.