

On strongly regular graphs with $m_2 = qm_3$ and $m_3 = qm_2$ for $q = 11, 12$

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Abstract: We say that a regular graph G of order n and degree $r \geq 1$ (which is not the complete graph) is strongly regular if there exist non-negative integers τ and θ such that $|S_i \cap S_j| = \tau$ for any two adjacent vertices i and j , and $|S_i \cap S_j| = \theta$ for any two distinct non-adjacent vertices i and j , where S_k denotes the neighborhood of the vertex k . Let $\lambda_1 = r$, λ_2 and λ_3 be the distinct eigenvalues of a connected strongly regular graph. Let $m_1 = 1$, m_2 and m_3 denote the multiplicity of r , λ_2 and λ_3 , respectively. We here describe the parameters n , r , τ and θ for strongly regular graphs with $m_2 = qm_3$ and $m_3 = qm_2$ for $q = 11, 12$.

Keywords: Strongly regular graph, conference graph, integral graph

1 Introduction

Let G be a simple graph of order n with vertex set $V(G) = \{1, 2, \dots, n\}$. The spectrum of G consists of the eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ of its (0,1) adjacency matrix A and is denoted by $\sigma(G)$. We say that a regular graph G of order n and degree $r \geq 1$ (which is not the complete graph K_n) is strongly regular if there exist non-negative integers τ and θ such that $|S_i \cap S_j| = \tau$ for any two adjacent vertices i and j , and $|S_i \cap S_j| = \theta$ for any two distinct non-adjacent vertices i and j , where $S_k \subseteq V(G)$ denotes the neighborhood of the vertex k . We know that a regular connected graph G is strongly regular if and only if it has exactly three distinct eigenvalues [1] (see also [3]). Let $\lambda_1 = r$, λ_2 and λ_3 denote the distinct eigenvalues of a connected strongly regular graph G . Let $m_1 = 1$, m_2 and m_3 denote the multiplicity of r , λ_2 and λ_3 . Further, let $\bar{r} = (n-1) - r$, $\bar{\lambda}_2 = -\lambda_3 - 1$ and $\bar{\lambda}_3 = -\lambda_2 - 1$ denote the distinct eigenvalues of the strongly regular graph \bar{G} , where \bar{G} denotes the complement of G . Then $\bar{\tau} = n - 2r - 2 + \theta$ and $\bar{\theta} = n - 2r + \tau$ where $\bar{\tau} = \tau(\bar{G})$ and $\bar{\theta} = \theta(\bar{G})$.

Remark 1 (i) if G is a disconnected strongly regular graph of degree r then $G = mK_{r+1}$, where mH denotes the m -fold union of the graph H ; (ii) G is a disconnected strongly regular graph if and only if $\theta = 0$.

Remark 2 (i) a strongly regular graph G of order $n = 4k + 1$ and degree $r = 2k$ with $\tau = k - 1$ and $\theta = k$ is called a conference graph; (ii) a strongly regular graph is a conference graph if and only if $m_2 = m_3$ and (iii) if $m_2 \neq m_3$ then G is an integral¹ graph.

We have recently started to investigate strongly regular graphs with $m_2 = qm_3$ and $m_3 = qm_2$, where q is a positive integer [4]. In the some work we have described the parameters n, r, τ and θ for strongly regular graphs with $m_2 = qm_3$ and $m_3 = qm_2$ for $q = 2, 3, 4$. In particular, (i) we have described in [5] the parameters n, r, τ and θ for strongly regular graphs with $m_2 = qm_3$ and $m_3 = qm_2$ for $q = 5, 6, 7, 8$ and (ii) we have described in [6] the parameters n, r, τ and θ for strongly regular graphs with $m_2 = qm_3$ and $m_3 = qm_2$ for $q = 9, 10$. We now proceed to establish the parameters of strongly regular graphs with $m_2 = qm_3$ and $m_3 = qm_2$ for $q = 11, 12$, as follows. First,

Proposition 1 (Elzinga [2]) *Let G be a connected or disconnected strongly regular graph of order n and degree r . Then*

$$r^2 - (\tau - \theta + 1)r - (n - 1)\theta = 0. \quad (1)$$

Proposition 2 (Elzinga [2]) *Let G be a connected strongly regular graph of order n and degree r . Then*

$$2r + (\tau - \theta)(m_2 + m_3) + \delta(m_2 - m_3) = 0, \quad (2)$$

where $\delta = \lambda_2 - \lambda_3$.

Remark 3 (Lepović [4]) *Using the same procedure applied in [4] we can establish the parameters n, r, τ and θ for strongly regular graphs with $m_2 = qm_3$ and $m_3 = qm_2$ for any fixed value $q \in \mathbb{N}$, as follows. First, let $m_3 = p, m_2 = qp$ and $n = (q + 1)p + 1$ where $q \in \mathbb{N}$. Using (2) we obtain $r = p(|\lambda_3| - q\lambda_2)$. Let $|\lambda_3| - q\lambda_2 = t$ where $t = 1, 2, \dots, q$. Let $\lambda_2 = k$ where k is a positive integer. Then (i) $\lambda_3 = -(qk + t)$; (ii) $\tau - \theta = -((q - 1)k + t)$; (iii) $\delta = (q + 1)k + t$; (iv) $r = pt$ and (v) $\theta = pt - qk^2 - kt$. Using (ii), (iv) and (v) we can easily see that (1) is reduced to*

$$(p + 1)t^2 - ((q + 1)p + 1)t + q(q + 1)k^2 + 2qkt = 0. \quad (3)$$

Second, let $m_2 = p, m_3 = qp$ and $n = (q + 1)p + 1$ where $q \in \mathbb{N}$. Using (2) we obtain $r = p(q|\lambda_3| - \lambda_2)$. Let $q|\lambda_3| - \lambda_2 = t$ where $t = 1, 2, \dots, q$. Let $\lambda_3 = -k$ where k is a

¹We say that a connected or disconnected graph G is integral if its spectrum $\sigma(G)$ consists only of integral values.

positive integer. Then (i) $\lambda_2 = qk - t$; (ii) $\tau - \theta = (q - 1)k - t$; (iii) $\delta = (q + 1)k - t$; (iv) $r = pt$ and (v) $\theta = pt - qk^2 + kt$. Using (ii), (iv) and (v) we can easily see that (1) is reduced to

$$(p + 1)t^2 - ((q + 1)p + 1)t + q(q + 1)k^2 - 2qkt = 0. \quad (4)$$

Using (3) and (4) we can obtain for $t = 1, 2, \dots, q$ the corresponding classes of strongly regular graphs with $m_2 = qm_3$ and $m_3 = qm_2$, respectively.

2 Main results

Remark 4 Since $m_2(\overline{G}) = m_3(G)$ and $m_3(\overline{G}) = m_2(G)$ we note that if $m_2(G) = qm_3(G)$ then $m_3(\overline{G}) = qm_2(\overline{G})$.

Remark 5 In Theorems 1 and 2 the complements of strongly regular graphs appear in pairs in (k^0) and (\overline{k}^0) classes, where k denotes the corresponding number of a class.

Remark 6 $\overline{\alpha K_\beta}$ is a strongly regular graph of order $n = \alpha\beta$ and degree $r = (\alpha - 1)\beta$ with $\tau = (\alpha - 2)\beta$ and $\theta = (\alpha - 1)\beta$. Its eigenvalues are $\lambda_2 = 0$ and $\lambda_3 = -\beta$ with $m_2 = \alpha(\beta - 1)$ and $m_3 = \alpha - 1$.

Proposition 3 Let G be a connected strongly regular graph of order n and degree r with $m_2 = 11m_3$. Then G belongs to the class $(\overline{2}^0)$ or (3^0) or (4^0) or $(\overline{5}^0)$ represented in Theorem 1.

Proof. Let $m_3 = p$, $m_2 = 11p$ and $n = 12p + 1$ where $p \in \mathbb{N}$. Using (2) we obtain $r = p(|\lambda_3| - 11\lambda_2)$. Let $|\lambda_3| - 11\lambda_2 = t$ where $t = 1, 2, \dots, 11$. Let $\lambda_2 = k$ where k is a positive integer. Then according to Remark 3 we have (i) $\lambda_3 = -(11k + t)$; (ii) $\tau - \theta = -(10k + t)$; (iii) $\delta = 12k + t$; (iv) $r = pt$ and (v) $\theta = pt - 11k^2 - kt$. In this case we can easily see that (3) is reduced to

$$(p + 1)t^2 - (12p + 1)t + 132k^2 + 22kt = 0. \quad (5)$$

Case 1. ($t = 1$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(11k + 1)$, $\tau - \theta = -(10k + 1)$, $\delta = 12k + 1$, $r = p$ and $\theta = p - 11k^2 - k$. Using (5) we find that $p = 2k(6k + 1)$. So we obtain that G is a strongly regular graph of order $n = (12k + 1)^2$ and degree $r = 2k(6k + 1)$ with $\tau = k^2 - 9k - 1$ and $\theta = k(k + 1)$.

Case 2. ($t = 2$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(11k + 2)$, $\tau - \theta = -(10k + 2)$, $\delta = 12k + 2$, $r = 2p$ and $\theta = 2p - 11k^2 - 2k$. Using (5) we find that $10p - 1 = 22k(3k + 1)$, a contradiction because $2 \nmid 10p - 1$.

Case 3. ($t = 3$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(11k + 3)$, $\tau - \theta = -(10k + 3)$, $\delta = 12k + 3$, $r = 3p$ and $\theta = 3p - 11k^2 - 3k$. Using (5) we find that $9p - 2 = 22k(2k + 1)$, a contradiction because $9 \nmid 44k^2 + 22k + 2$.

Case 4. ($t = 4$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(11k + 4)$, $\tau - \theta = -(10k + 4)$, $\delta = 12k + 4$, $r = 4p$ and $\theta = 4p - 11k^2 - 4k$. Using (5) we find that $8p - 3 = 11k(3k + 2)$, a contradiction because $8 \nmid 33k^2 + 22k + 3$.

Case 5. ($t = 5$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(11k + 5)$, $\tau - \theta = -(10k + 5)$, $\delta = 12k + 5$, $r = 5p$ and $\theta = 5p - 11k^2 - 5k$. Using (5) we find that $35p - 20 = 22k(6k + 5)$. Replacing k with $35k - 15$ we arrive at $p = 4620k^2 - 3850k + 802$. So we obtain that G is a strongly regular graph of order $n = 385(12k - 5)^2$ and degree $r = 5(4620k^2 - 3850k + 802)$ with $\tau = 5(1925k^2 - 1645k + 351)$ and $\theta = 35(5k - 2)(55k - 23)$.

Case 6. ($t = 6$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(11k + 6)$, $\tau - \theta = -(10k + 6)$, $\delta = 12k + 6$, $r = 6p$ and $\theta = 6p - 11k^2 - 6k$. Using (5) we find that $6p - 5 = 22k(k + 1)$, a contradiction because $2 \nmid 6p - 5$.

Case 7. ($t = 7$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(11k + 7)$, $\tau - \theta = -(10k + 7)$, $\delta = 12k + 7$, $r = 7p$ and $\theta = 7p - 11k^2 - 7k$. Using (5) we find that $35p - 42 = 22k(6k + 7)$. Replacing k with $35k + 14$ we arrive at $p = 4620k^2 + 3850k + 802$. So we obtain that G is a strongly regular graph of order $n = 385(12k + 5)^2$ and degree $r = 7(4620k^2 + 3850k + 802)$ with $\tau = 7(2695k^2 + 2225k + 459)$ and $\theta = 35(7k + 3)(77k + 32)$.

Case 8. ($t = 8$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(11k + 8)$, $\tau - \theta = -(10k + 8)$, $\delta = 12k + 8$, $r = 8p$ and $\theta = 8p - 11k^2 - 8k$. Using (5) we find that $8p - 14 = 11k(3k + 4)$, a contradiction because $8 \nmid 33k^2 + 44k + 14$.

Case 9. ($t = 9$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(11k + 9)$, $\tau - \theta = -(10k + 9)$, $\delta = 12k + 9$, $r = 9p$ and $\theta = 9p - 11k^2 - 9k$. Using (5) we find that $9p - 24 = 22k(2k + 3)$, a contradiction because $9 \nmid 44k^2 + 66k + 24$.

Case 10. ($t = 10$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(11k + 10)$, $\tau - \theta = -(10k + 10)$, $\delta = 12k + 10$, $r = 10p$ and $\theta = 10p - 11k^2 - 10k$. Using (5) we find that $10p - 45 = 22k(3k + 5)$, a contradiction because $2 \nmid 10p - 45$.

Case 11. ($t = 11$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(11k + 11)$, $\tau - \theta = -(10k + 11)$, $\delta = 12k + 11$, $r = 11p$ and $\theta = 11p - 11k^2 - 11k$. Using (5) we find that $p = 2(k + 1)(6k + 5)$. Replacing k with $k - 1$ we arrive at $p = 2k(6k - 1)$. So we obtain that G is a strongly regular graph of order $n = (12k - 1)^2$ and degree $r = 22k(6k - 1)$ with $\tau = 121k^2 - 21k - 1$ and $\theta = 11k(11k - 1)$. \square

Proposition 4 *Let G be a connected strongly regular graph of order n and degree r with $m_3 = 11m_2$. Then G belongs to the class (2^0) or $(\bar{3}^0)$ or $(\bar{4}^0)$ or (5^0) represented in Theorem 1.*

Proof. Let $m_2 = p$, $m_3 = 11p$ and $n = 12p + 1$ where $p \in \mathbb{N}$. Using (2) we obtain $r = p(11|\lambda_3| - \lambda_2)$. Let $11|\lambda_3| - \lambda_2 = t$ where $t = 1, 2, \dots, 11$. Let $\lambda_3 = -k$ where k is a positive integer. Then according to Remark 3 we have (i) $\lambda_2 = 11k - t$; (ii) $\tau - \theta = 10k - t$;

(iii) $\delta = 12k - t$; (iv) $r = pt$ and (v) $\theta = pt - 11k^2 + kt$. In this case we can easily see that (4) is reduced to

$$(p+1)t^2 - (12p+1)t + 132k^2 - 22kt = 0. \quad (6)$$

Case 1. ($t = 1$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 11k - 1$ and $\lambda_3 = -k$, $\tau - \theta = 10k - 1$, $\delta = 12k - 1$, $r = p$ and $\theta = p - 11k^2 + k$. Using (6) we find that $p = 2k(6k - 1)$. So we obtain that G is a strongly regular graph of order $n = (12k - 1)^2$ and degree $r = 2k(6k - 1)$ with $\tau = k^2 + 9k - 1$ and $\theta = k(k - 1)$.

Case 2. ($t = 2$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 11k - 2$ and $\lambda_3 = -k$, $\tau - \theta = 10k - 2$, $\delta = 12k - 2$, $r = 2p$ and $\theta = 2p - 11k^2 + 2k$. Using (6) we find that $10p - 1 = 22k(3k - 1)$, a contradiction because $2 \nmid 10p - 1$.

Case 3. ($t = 3$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 11k - 3$ and $\lambda_3 = -k$, $\tau - \theta = 10k - 3$, $\delta = 12k - 3$, $r = 3p$ and $\theta = 3p - 11k^2 + 3k$. Using (6) we find that $9p - 2 = 22k(2k - 1)$, a contradiction because $9 \nmid 44k^2 - 22k + 2$.

Case 4. ($t = 4$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 11k - 4$ and $\lambda_3 = -k$, $\tau - \theta = 10k - 4$, $\delta = 12k - 4$, $r = 4p$ and $\theta = 4p - 11k^2 + 4k$. Using (6) we find that $8p - 3 = 11k(3k - 2)$, a contradiction because $8 \nmid 33k^2 - 22k + 3$.

Case 5. ($t = 5$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 11k - 5$ and $\lambda_3 = -k$, $\tau - \theta = 10k - 5$, $\delta = 12k - 5$, $r = 5p$ and $\theta = 5p - 11k^2 + 5k$. Using (6) we find that $35p - 20 = 22k(6k - 5)$. Replacing k with $35k + 15$ we arrive at $p = 4620k^2 + 3850k + 802$. So we obtain that G is a strongly regular graph of order $n = 385(12k + 5)^2$ and degree $r = 5(4620k^2 + 3850k + 802)$ with $\tau = 5(1925k^2 + 1645k + 351)$ and $\theta = 35(5k + 2)(55k + 23)$.

Case 6. ($t = 6$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 11k - 6$ and $\lambda_3 = -k$, $\tau - \theta = 10k - 6$, $\delta = 12k - 6$, $r = 6p$ and $\theta = 6p - 11k^2 + 6k$. Using (6) we find that $6p - 5 = 22k(k - 1)$, a contradiction because $2 \nmid 6p - 1$.

Case 7. ($t = 7$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 11k - 7$ and $\lambda_3 = -k$, $\tau - \theta = 10k - 7$, $\delta = 12k - 7$, $r = 7p$ and $\theta = 7p - 11k^2 + 7k$. Using (6) we find that $35p - 42 = 22k(6k - 7)$. Replacing k with $35k - 14$ we arrive at $p = 4620k^2 - 3850k + 802$. So we obtain that G is a strongly regular graph of order $n = 385(12k - 5)^2$ and degree $r = 7(4620k^2 - 3850k + 802)$ with $\tau = 7(2695k^2 - 2225k + 459)$ and $\theta = 35(7k - 3)(77k - 32)$.

Case 8. ($t = 8$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 11k - 8$ and $\lambda_3 = -k$, $\tau - \theta = 10k - 8$, $\delta = 12k - 8$, $r = 8p$ and $\theta = 8p - 11k^2 + 8k$. Using (6) we find that $8p - 14 = 11k(3k - 4)$, a contradiction because $8 \nmid 33k^2 - 44k + 14$.

Case 9. ($t = 9$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 11k - 9$ and $\lambda_3 = -k$, $\tau - \theta = 10k - 9$, $\delta = 12k - 9$, $r = 9p$ and $\theta = 9p - 11k^2 + 9k$. Using (6) we find that $9p - 24 = 22k(2k - 3)$, a contradiction because $9 \nmid 44k^2 - 66k + 24$.

Case 10. ($t = 10$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 11k - 10$ and $\lambda_3 = -k$, $\tau - \theta = 10k - 10$, $\delta = 12k - 10$, $r = 10p$ and $\theta = 10p - 11k^2 + 10k$. Using (6) we find that $10p - 45 = 22k(3k - 5)$, a contradiction because $2 \nmid 10p - 45$.

Case 11. ($t = 11$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 11k - 11$ and $\lambda_3 = -k$, $\tau - \theta = 10k - 11$, $\delta = 12k - 11$, $r = 11p$ and $\theta = 11p - 11k^2 + 11k$. Using (6) we find that $p = 2(k - 1)(6k - 5)$. Replacing k with $k + 1$ we arrive at $p = 2k(6k + 1)$. So we obtain that G is a strongly regular graph of order $n = (12k + 1)^2$ and degree $r = 22k(6k + 1)$ with $\tau = 121k^2 + 21k - 1$ and $\theta = 11k(11k + 1)$. \square

Remark 7 We note that $\overline{11K_{11}}$ is a strongly regular graph with $m_2 = 11m_3$. It is obtained from the class Theorem 1 ($\bar{2}^0$) for $k = 1$.

Theorem 1 Let G be a connected strongly regular graph of order n and degree r with $m_2 = 11m_3$ or $m_3 = 11m_2$. Then G is one of the following strongly regular graphs:

- (1^0) G is the strongly regular graph $\overline{11K_{11}}$ of order $n = 121$ and degree $r = 110$ with $\tau = 99$ and $\theta = 110$. Its eigenvalues are $\lambda_2 = 0$ and $\lambda_3 = -11$ with $m_2 = 110$ and $m_3 = 10$;
- (2^0) G is a strongly regular graph of order $n = (12k - 1)^2$ and degree $r = 2k(6k - 1)$ with $\tau = k^2 + 9k - 1$ and $\theta = k(k - 1)$, where $k \geq 2$. Its eigenvalues are $\lambda_2 = 11k - 1$ and $\lambda_3 = -k$ with $m_2 = 2k(6k - 1)$ and $m_3 = 22k(6k - 1)$;
- ($\bar{2}^0$) G is a strongly regular graph of order $n = (12k - 1)^2$ and degree $r = 22k(6k - 1)$ with $\tau = 121k^2 - 21k - 1$ and $\theta = 11k(11k - 1)$, where $k \geq 2$. Its eigenvalues are $\lambda_2 = k - 1$ and $\lambda_3 = -11k$ with $m_2 = 22k(6k - 1)$ and $m_3 = 2k(6k - 1)$;
- (3^0) G is a strongly regular graph of order $n = (12k + 1)^2$ and degree $r = 2k(6k + 1)$ with $\tau = k^2 - 9k - 1$ and $\theta = k(k + 1)$, where $k \geq 10$. Its eigenvalues are $\lambda_2 = k$ and $\lambda_3 = -(11k + 1)$ with $m_2 = 22k(6k + 1)$ and $m_3 = 2k(6k + 1)$;
- ($\bar{3}^0$) G is a strongly regular graph of order $n = (12k + 1)^2$ and degree $r = 22k(6k + 1)$ with $\tau = 121k^2 + 21k - 1$ and $\theta = 11k(11k + 1)$, where $k \geq 10$. Its eigenvalues are $\lambda_2 = 11k$ and $\lambda_3 = -(k + 1)$ with $m_2 = 2k(6k + 1)$ and $m_3 = 22k(6k + 1)$;
- (4^0) G is a strongly regular graph of order $n = 385(12k - 5)^2$ and degree $r = 5(4620k^2 - 3850k + 802)$ with $\tau = 5(1925k^2 - 1645k + 351)$ and $\theta = 35(5k - 2)(55k - 23)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 35k - 15$ and $\lambda_3 = -(385k - 160)$ with $m_2 = 11(4620k^2 - 3850k + 802)$ and $m_3 = 4620k^2 - 3850k + 802$;
- ($\bar{4}^0$) G is a strongly regular graph of order $n = 385(12k - 5)^2$ and degree $r = 7(4620k^2 - 3850k + 802)$ with $\tau = 7(2695k^2 - 2225k + 459)$ and $\theta = 35(7k - 3)(77k - 32)$,

where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 385k - 161$ and $\lambda_3 = -(35k - 14)$ with $m_2 = 4620k^2 - 3850k + 802$ and $m_3 = 11(4620k^2 - 3850k + 802)$;

(5^0) G is a strongly regular graph of order $n = 385(12k + 5)^2$ and degree $r = 5(4620k^2 + 3850k + 802)$ with $\tau = 5(1925k^2 + 1645k + 351)$ and $\theta = 35(5k + 2)(55k + 23)$, where $k \geq 0$. Its eigenvalues are $\lambda_2 = 385k + 160$ and $\lambda_3 = -(35k + 15)$ with $m_2 = 4620k^2 + 3850k + 802$ and $m_3 = 11(4620k^2 + 3850k + 802)$;

($\bar{5}^0$) G is a strongly regular graph of order $n = 385(12k + 5)^2$ and degree $r = 7(4620k^2 + 3850k + 802)$ with $\tau = 7(2695k^2 + 2225k + 459)$ and $\theta = 35(7k + 3)(77k + 32)$, where $k \geq 0$. Its eigenvalues are $\lambda_2 = 35k + 14$ and $\lambda_3 = -(385k + 161)$ with $m_2 = 11(4620k^2 + 3850k + 802)$ and $m_3 = 4620k^2 + 3850k + 802$;

Proof. First, according to Remark 6 we have $\alpha(\beta - 1) = 11(\alpha - 1)$, from which we find that $\alpha = 11$, $\beta = 11$. In view of this we obtain the strongly regular graph represented in Theorem 1 (1^0). Next, according to Proposition 3 it turns out that G belongs to the class ($\bar{2}^0$) or (3^0) or (4^0) or ($\bar{5}^0$) if $m_2 = 11m_3$. According to Proposition 4 it turns out that G belongs to the class (2^0) or ($\bar{3}^0$) or ($\bar{4}^0$) or (5^0) if $m_3 = 11m_2$. \square

Proposition 5 Let G be a connected strongly regular graph of order n and degree r with $m_2 = 12m_3$. Then G belongs to the class ($\bar{6}^0$) or (7^0) or (8^0) or ($\bar{9}^0$) or (10^0) or ($\bar{11}^0$) or (12^0) or ($\bar{13}^0$) or (14^0) or ($\bar{15}^0$) or (16^0) or ($\bar{17}^0$) represented in Theorem 2.

Proof. Let $m_3 = p$, $m_2 = 12p$ and $n = 13p + 1$ where $p \in \mathbb{N}$. Using (2) we obtain $r = p(|\lambda_3| - 12\lambda_2)$. Let $|\lambda_3| - 12\lambda_2 = t$ where $t = 1, 2, \dots, 12$. Let $\lambda_2 = k$ where k is a positive integer. Then according to Remark 3 we have (i) $\lambda_3 = -(12k + t)$; (ii) $\tau - \theta = -(11k + t)$; (iii) $\delta = 13k + t$; (iv) $r = pt$ and (v) $\theta = pt - 12k^2 - kt$. In this case we can easily see that (3) is reduced to

$$(p + 1)t^2 - (13p + 1)t + 156k^2 + 24kt = 0. \quad (7)$$

Case 1. ($t = 1$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(12k + 1)$, $\tau - \theta = -(11k + 1)$, $\delta = 13k + 1$, $r = p$ and $\theta = p - 12k^2 - k$. Using (7) we find that $p = k(13k + 2)$. So we obtain that G is a strongly regular graph of order $n = (13k + 1)^2$ and degree $r = k(13k + 2)$ with $\tau = k^2 - 10k - 1$ and $\theta = k(k + 1)$.

Case 2. ($t = 2$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(12k + 2)$, $\tau - \theta = -(11k + 2)$, $\delta = 13k + 2$, $r = 2p$ and $\theta = 2p - 12k^2 - 2k$. Using (7) we find that $11p - 1 = 6k(13k + 4)$. Replacing k with $11k - 1$ we arrive at $p = 858k^2 - 132k + 5$. So we obtain that G is a strongly regular graph of order $n = 66(13k - 1)^2$ and degree $r = 2(858k^2 - 132k + 5)$ with $\tau = 264k^2 - 143k + 9$ and $\theta = 22k(12k - 1)$.

Case 3. ($t = 3$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(12k + 3)$, $\tau - \theta = -(11k + 3)$, $\delta = 13k + 3$, $r = 3p$ and $\theta = 3p - 12k^2 - 3k$. Using (7) we find that $5p - 1 = 2k(13k + 6)$. Replacing k with $5k - 1$ we arrive at $p = 130k^2 - 40k + 3$. So we obtain that G is a strongly regular graph of order $n = 10(13k - 2)^2$ and degree $r = 3(130k^2 - 40k + 3)$ with $\tau = 90k^2 - 70k + 8$ and $\theta = 15k(6k - 1)$.

Case 4. ($t = 4$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(12k + 4)$, $\tau - \theta = -(11k + 4)$, $\delta = 13k + 4$, $r = 4p$ and $\theta = 4p - 12k^2 - 4k$. Using (7) we find that $3p - 1 = k(13k + 8)$. Replacing k with $3k - 1$ we arrive at $p = 39k^2 - 18k + 2$. So we obtain that G is a strongly regular graph of order $n = 3(13k - 3)^2$ and degree $r = 4(39k^2 - 18k + 2)$ with $\tau = 48k^2 - 45k + 7$ and $\theta = 12k(4k - 1)$.

Case 5. ($t = 5$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(12k + 5)$, $\tau - \theta = -(11k + 5)$, $\delta = 13k + 5$, $r = 5p$ and $\theta = 5p - 12k^2 - 5k$. Using (7) we find that $10p - 5 = 3k(13k + 10)$. Replacing k with $10k - 5$ we arrive at $p = 390k^2 - 360k + 83$. So we obtain that G is a strongly regular graph of order $n = 30(13k - 6)^2$ and degree $r = 5(390k^2 - 360k + 83)$ with $\tau = 10(75k^2 - 76k + 19)$ and $\theta = 10(5k - 2)(15k - 7)$.

Case 6. ($t = 6$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(12k + 6)$, $\tau - \theta = -(11k + 6)$, $\delta = 13k + 6$, $r = 6p$ and $\theta = 6p - 12k^2 - 6k$. Using (7) we find that $7p - 5 = 2k(13k + 12)$. Replacing k with $7k - 1$ we arrive at $p = 182k^2 - 28k + 1$. So we obtain that G is a strongly regular graph of order $n = 14(13k - 1)^2$ and degree $r = 6(182k^2 - 28k + 1)$ with $\tau = 504k^2 - 119k + 5$ and $\theta = 42k(12k - 1)$.

Case 7. ($t = 7$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(12k + 7)$, $\tau - \theta = -(11k + 7)$, $\delta = 13k + 7$, $r = 7p$ and $\theta = 7p - 12k^2 - 7k$. Using (7) we find that $7p - 7 = 2k(13k + 14)$. Replacing k with $7k$ we arrive at $p = 182k^2 + 28k + 1$. So we obtain that G is a strongly regular graph of order $n = 14(13k + 1)^2$ and degree $r = 7(182k^2 + 28k + 1)$ with $\tau = 14k(49k + 5)$ and $\theta = 7(7k + 1)(14k + 1)$.

Case 8. ($t = 8$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(12k + 8)$, $\tau - \theta = -(11k + 8)$, $\delta = 13k + 8$, $r = 8p$ and $\theta = 8p - 12k^2 - 8k$. Using (7) we find that $10p - 14 = 3k(13k + 16)$. Replacing k with $10k + 4$ we arrive at $p = 390k^2 + 360k + 83$. So we obtain that G is a strongly regular graph of order $n = 30(13k + 6)^2$ and degree $r = 8(390k^2 + 360k + 83)$ with $\tau = 2(960k^2 + 865k + 194)$ and $\theta = 40(2k + 1)(24k + 11)$.

Case 9. ($t = 9$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(12k + 9)$, $\tau - \theta = -(11k + 9)$, $\delta = 13k + 9$, $r = 9p$ and $\theta = 9p - 12k^2 - 9k$. Using (7) we find that $3p - 6 = k(13k + 18)$. Replacing k with $3k$ we arrive at $p = 39k^2 + 18k + 2$. So we obtain that G is a strongly regular graph of order $n = 3(13k + 3)^2$ and degree $r = 9(39k^2 + 18k + 2)$ with $\tau = 3(81k^2 + 34k + 3)$ and $\theta = 9(3k + 1)(9k + 2)$.

Case 10. ($t = 10$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(12k + 10)$, $\tau - \theta = -(11k + 10)$, $\delta = 13k + 10$, $r = 10p$ and $\theta = 10p - 12k^2 - 10k$. Using (7) we find

that $5p - 15 = 2k(13k + 20)$. Replacing k with $5k$ we arrive at $p = 130k^2 + 40k + 3$. So we obtain that G is a strongly regular graph of order $n = 10(13k + 2)^2$ and degree $r = 10(130k^2 + 40k + 3)$ with $\tau = 5(200k^2 + 59k + 4)$ and $\theta = 10(5k + 1)(20k + 3)$.

Case 11. ($t = 11$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(12k + 11)$, $\tau - \theta = -(11k + 11)$, $\delta = 13k + 11$, $r = 11p$ and $\theta = 11p - 12k^2 - 11k$. Using (7) we find that $11p - 55 = 6k(13k + 22)$. Replacing k with $11k$ we arrive at $p = 858k^2 + 132k + 5$. So we obtain that G is a strongly regular graph of order $n = 66(13k + 1)^2$ and degree $r = 11(858k^2 + 132k + 5)$ with $\tau = 22(11k + 1)(33k + 2)$ and $\theta = 11(11k + 1)(66k + 5)$.

Case 12. ($t = 12$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(12k + 12)$, $\tau - \theta = -(11k + 12)$, $\delta = 13k + 12$, $r = 12p$ and $\theta = 12p - 12k^2 - 12k$. Using (7) we find that $p = (k + 1)(13k + 11)$. Replacing k with $k - 1$ we arrive at $p = k(13k - 2)$. So we obtain that G is a strongly regular graph of order $n = (13k - 1)^2$ and degree $r = 12k(13k - 2)$ with $\tau = 144k^2 - 23k - 1$ and $\theta = 12k(12k - 1)$. \square

Proposition 6 *Let G be a connected strongly regular graph of order n and degree r with $m_3 = 12m_2$. Then G belongs to the class (6^0) or (7^0) or (8^0) or (9^0) or (10^0) or (11^0) or (12^0) or (13^0) or (14^0) or (15^0) or (16^0) or (17^0) represented in Theorem 2.*

Proof. Let $m_2 = p$, $m_3 = 12p$ and $n = 13p + 1$ where $p \in \mathbb{N}$. Using (2) we obtain $r = p(12|\lambda_3| - \lambda_2)$. Let $12|\lambda_3| - \lambda_2 = t$ where $t = 1, 2, \dots, 12$. Let $\lambda_3 = -k$ where k is a positive integer. Then according to Remark 3 we have (i) $\lambda_2 = 12k - t$; (ii) $\tau - \theta = 11k - t$; (iii) $\delta = 13k - t$; (iv) $r = pt$ and (v) $\theta = pt - 12k^2 + kt$. In this case we can easily see that (4) is reduced to

$$(p + 1)t^2 - (13p + 1)t + 156k^2 - 24kt = 0. \quad (8)$$

Case 1. ($t = 1$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 12k - 1$ and $\lambda_3 = -k$, $\tau - \theta = 11k - 1$, $\delta = 13k - 1$, $r = p$ and $\theta = p - 12k^2 + k$. Using (8) we find that $p = k(13k - 2)$. So we obtain that G is a strongly regular graph of order $n = (13k - 1)^2$ and degree $r = k(13k - 2)$ with $\tau = k^2 + 10k - 1$ and $\theta = k(k - 1)$.

Case 2. ($t = 2$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 12k - 2$ and $\lambda_3 = -k$, $\tau - \theta = 11k - 2$, $\delta = 13k - 2$, $r = 2p$ and $\theta = 2p - 12k^2 + 2k$. Using (8) we find that $11p - 1 = 6k(13k - 4)$. Replacing k with $11k + 1$ we arrive at $p = 858k^2 + 132k + 5$. So we obtain that G is a strongly regular graph of order $n = 66(13k + 1)^2$ and degree $r = 2(858k^2 + 132k + 5)$ with $\tau = 264k^2 + 143k + 9$ and $\theta = 22k(12k + 1)$.

Case 3. ($t = 3$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 12k - 3$ and $\lambda_3 = -k$, $\tau - \theta = 11k - 3$, $\delta = 13k - 3$, $r = 3p$ and $\theta = 3p - 12k^2 + 3k$. Using (8) we find that $5p - 1 = 2k(13k - 6)$. Replacing k with $5k + 1$ we arrive at $p = 130k^2 + 40k + 3$. So we obtain that G is a strongly regular graph of order $n = 10(13k + 2)^2$ and degree $r = 3(130k^2 + 40k + 3)$ with $\tau = 90k^2 + 70k + 8$ and $\theta = 15k(6k + 1)$.

Case 4. ($t = 4$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 12k - 4$ and $\lambda_3 = -k$, $\tau - \theta = 11k - 4$, $\delta = 13k - 4$, $r = 4p$ and $\theta = 4p - 12k^2 + 4k$. Using (8) we find that $3p - 1 = k(13k - 8)$. Replacing k with $3k + 1$ we arrive at $p = 39k^2 + 18k + 2$. So we obtain that G is a strongly regular graph of order $n = 3(13k + 3)^2$ and degree $r = 4(39k^2 + 18k + 2)$ with $\tau = 48k^2 + 45k + 7$ and $\theta = 12k(4k + 1)$.

Case 5. ($t = 5$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 12k - 5$ and $\lambda_3 = -k$, $\tau - \theta = 11k - 5$, $\delta = 13k - 5$, $r = 5p$ and $\theta = 5p - 12k^2 + 5k$. Using (8) we find that $10p - 5 = 3k(13k - 10)$. Replacing k with $10k + 5$ we arrive at $p = 390k^2 + 360k + 83$. So we obtain that G is a strongly regular graph of order $n = 30(13k + 6)^2$ and degree $r = 5(390k^2 + 360k + 83)$ with $\tau = 10(75k^2 + 76k + 19)$ and $\theta = 10(5k + 2)(15k + 7)$.

Case 6. ($t = 6$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 12k - 6$ and $\lambda_3 = -k$, $\tau - \theta = 11k - 6$, $\delta = 13k - 6$, $r = 6p$ and $\theta = 6p - 12k^2 + 6k$. Using (8) we find that $7p - 5 = 2k(13k - 12)$. Replacing k with $7k + 1$ we arrive at $p = 182k^2 + 28k + 1$. So we obtain that G is a strongly regular graph of order $n = 14(13k + 1)^2$ and degree $r = 6(182k^2 + 28k + 1)$ with $\tau = 504k^2 + 119k + 5$ and $\theta = 42k(12k + 1)$.

Case 7. ($t = 7$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 12k - 7$ and $\lambda_3 = -k$, $\tau - \theta = 11k - 7$, $\delta = 13k - 7$, $r = 7p$ and $\theta = 7p - 12k^2 + 7k$. Using (8) we find that $7p - 7 = 2k(13k - 14)$. Replacing k with $7k$ we arrive at $p = 182k^2 - 28k + 1$. So we obtain that G is a strongly regular graph of order $n = 14(13k - 1)^2$ and degree $r = 7(182k^2 - 28k + 1)$ with $\tau = 14k(49k - 5)$ and $\theta = 7(7k - 1)(14k - 1)$.

Case 8. ($t = 8$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 12k - 8$ and $\lambda_3 = -k$, $\tau - \theta = 11k - 8$, $\delta = 13k - 8$, $r = 8p$ and $\theta = 8p - 12k^2 + 8k$. Using (8) we find that $10p - 14 = 3k(13k - 16)$. Replacing k with $10k - 4$ we arrive at $p = 390k^2 - 360k + 83$. So we obtain that G is a strongly regular graph of order $n = 30(13k - 6)^2$ and degree $r = 8(390k^2 - 360k + 83)$ with $\tau = 2(960k^2 - 865k + 194)$ and $\theta = 40(2k - 1)(24k - 11)$.

Case 9. ($t = 9$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 12k - 9$ and $\lambda_3 = -k$, $\tau - \theta = 11k - 9$, $\delta = 13k - 9$, $r = 9p$ and $\theta = 9p - 12k^2 + 9k$. Using (8) we find that $3p - 6 = k(13k - 18)$. Replacing k with $3k$ we arrive at $p = 39k^2 - 18k + 2$. So we obtain that G is a strongly regular graph of order $n = 3(13k - 3)^2$ and degree $r = 9(39k^2 - 18k + 2)$ with $\tau = 3(81k^2 - 34k + 3)$ and $\theta = 9(3k - 1)(9k - 2)$.

Case 10. ($t = 10$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 12k - 10$ and $\lambda_3 = -k$, $\tau - \theta = 11k - 10$, $\delta = 13k - 10$, $r = 10p$ and $\theta = 10p - 12k^2 + 10k$. Using (8) we find that $5p - 15 = 2k(13k - 20)$. Replacing k with $5k$ we arrive at $p = 130k^2 - 40k + 3$. So we obtain that G is a strongly regular graph of order $n = 10(13k - 2)^2$ and degree $r = 10(130k^2 - 40k + 3)$ with $\tau = 5(200k^2 - 59k + 4)$ and $\theta = 10(5k - 1)(20k - 3)$.

Case 11. ($t = 11$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 12k - 11$ and $\lambda_3 = -k$, $\tau - \theta = 11k - 11$, $\delta = 13k - 11$, $r = 11p$ and $\theta = 11p - 12k^2 + 11k$. Using (8) we find

that $11p - 55 = 6k(13k - 22)$. Replacing k with $11k$ we arrive at $p = 858k^2 - 132k + 5$. So we obtain that G is a strongly regular graph of order $n = 66(13k - 1)^2$ and degree $r = 11(858k^2 - 132k + 5)$ with $\tau = 22(11k - 1)(33k - 2)$ and $\theta = 11(11k - 1)(66k - 5)$.

Case 12. ($t = 12$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 12k - 12$ and $\lambda_3 = -k$, $\tau - \theta = 11k - 12$, $\delta = 13k - 12$, $r = 12p$ and $\theta = 12p - 12k^2 + 12k$. Using (8) we find that $p = (k - 1)(13k - 11)$. Replacing k with $k + 1$ we arrive at $p = k(13k + 2)$. So we obtain that G is a strongly regular graph of order $n = (13k + 1)^2$ and degree $r = 12k(13k + 2)$ with $\tau = 144k^2 + 23k - 1$ and $\theta = 12k(12k + 1)$. \square

Remark 8 We note that the complete bipartite graph $K_{7,7}$ is a strongly regular graph with $m_2 = 12m_3$. It is obtained from the class Theorem 2 $(\overline{13^0})$ for $k = 0$.

Remark 9 We note that $\overline{3K_9}$ is a strongly regular graph with $m_2 = 12m_3$. It is obtained from the class Theorem 2 $(\overline{9^0})$ for $k = 0$.

Remark 10 We note that $\overline{4K_{10}}$ is a strongly regular graph with $m_2 = 12m_3$. It is obtained from the class Theorem 2 $(\overline{11^0})$ for $k = 0$.

Remark 11 We note that $\overline{6K_{11}}$ is a strongly regular graph with $m_2 = 12m_3$. It is obtained from the class Theorem 2 $(\overline{17^0})$ for $k = 0$.

Remark 12 We note that $\overline{12K_{12}}$ is a strongly regular graph with $m_2 = 12m_3$. It is obtained from the class Theorem 2 $(\overline{6^0})$ for $k = 1$.

Theorem 2 Let G be a connected strongly regular graph of order n and degree r with $m_2 = 12m_3$ or $m_3 = 12m_2$. Then G is one of the following strongly regular graphs:

- (1⁰) G is the complete bipartite graph $K_{7,7}$ of order $n = 14$ and degree $r = 7$ with $\tau = 0$ and $\theta = 7$. Its eigenvalues are $\lambda_2 = 0$ and $\lambda_3 = -7$ with $m_2 = 12$ and $m_3 = 1$;
- (2⁰) G is the strongly regular graph $\overline{3K_9}$ of order $n = 27$ and degree $r = 18$ with $\tau = 9$ and $\theta = 18$. Its eigenvalues are $\lambda_2 = 0$ and $\lambda_3 = -9$ with $m_2 = 24$ and $m_3 = 2$;
- (3⁰) G is the strongly regular graph $\overline{4K_{10}}$ of order $n = 40$ and degree $r = 30$ with $\tau = 20$ and $\theta = 30$. Its eigenvalues are $\lambda_2 = 0$ and $\lambda_3 = -10$ with $m_2 = 36$ and $m_3 = 3$;
- (4⁰) G is the strongly regular graph $\overline{6K_{11}}$ of order $n = 66$ and degree $r = 55$ with $\tau = 44$ and $\theta = 55$. Its eigenvalues are $\lambda_2 = 0$ and $\lambda_3 = -11$ with $m_2 = 60$ and $m_3 = 5$;
- (5⁰) G is the strongly regular graph $\overline{12K_{12}}$ of order $n = 144$ and degree $r = 132$ with $\tau = 120$ and $\theta = 132$. Its eigenvalues are $\lambda_2 = 0$ and $\lambda_3 = -12$ with $m_2 = 132$ and $m_3 = 11$;

- (6⁰) G is a strongly regular graph of order $n = (13k - 1)^2$ and degree $r = k(13k - 2)$ with $\tau = k^2 + 10k - 1$ and $\theta = k(k - 1)$, where $k \geq 2$. Its eigenvalues are $\lambda_2 = 12k - 1$ and $\lambda_3 = -k$ with $m_2 = k(13k - 2)$ and $m_3 = 12k(13k - 2)$;
- ($\bar{6}$ ⁰) G is a strongly regular graph of order $n = (13k - 1)^2$ and degree $r = 12k(13k - 2)$ with $\tau = 144k^2 - 23k - 1$ and $\theta = 12k(12k - 1)$, where $k \geq 2$. Its eigenvalues are $\lambda_2 = k - 1$ and $\lambda_3 = -12k$ with $m_2 = 12k(13k - 2)$ and $m_3 = k(13k - 2)$;
- (7⁰) G is a strongly regular graph of order $n = (13k + 1)^2$ and degree $r = k(13k + 2)$ with $\tau = k^2 - 10k - 1$ and $\theta = k(k + 1)$, where $k \geq 11$. Its eigenvalues are $\lambda_2 = k$ and $\lambda_3 = -(12k + 1)$ with $m_2 = 12k(13k + 2)$ and $m_3 = k(13k + 2)$;
- ($\bar{7}$ ⁰) G is a strongly regular graph of order $n = (13k + 1)^2$ and degree $r = 12k(13k + 2)$ with $\tau = 144k^2 + 23k - 1$ and $\theta = 12k(12k + 1)$, where $k \geq 11$. Its eigenvalues are $\lambda_2 = 12k$ and $\lambda_3 = -(k + 1)$ with $m_2 = k(13k + 2)$ and $m_3 = 12k(13k + 2)$;
- (8⁰) G is a strongly regular graph of order $n = 3(13k - 3)^2$ and degree $r = 4(39k^2 - 18k + 2)$ with $\tau = 48k^2 - 45k + 7$ and $\theta = 12k(4k - 1)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 3k - 1$ and $\lambda_3 = -(36k - 8)$ with $m_2 = 12(39k^2 - 18k + 2)$ and $m_3 = 39k^2 - 18k + 2$;
- ($\bar{8}$ ⁰) G is a strongly regular graph of order $n = 3(13k - 3)^2$ and degree $r = 9(39k^2 - 18k + 2)$ with $\tau = 3(81k^2 - 34k + 3)$ and $\theta = 9(3k - 1)(9k - 2)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 36k - 9$ and $\lambda_3 = -3k$ with $m_2 = 39k^2 - 18k + 2$ and $m_3 = 12(39k^2 - 18k + 2)$;
- (9⁰) G is a strongly regular graph of order $n = 3(13k + 3)^2$ and degree $r = 4(39k^2 + 18k + 2)$ with $\tau = 48k^2 + 45k + 7$ and $\theta = 12k(4k + 1)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 36k + 8$ and $\lambda_3 = -(3k + 1)$ with $m_2 = 39k^2 + 18k + 2$ and $m_3 = 12(39k^2 + 18k + 2)$;
- ($\bar{9}$ ⁰) G is a strongly regular graph of order $n = 3(13k + 3)^2$ and degree $r = 9(39k^2 + 18k + 2)$ with $\tau = 3(81k^2 + 34k + 3)$ and $\theta = 9(3k + 1)(9k + 2)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 3k$ and $\lambda_3 = -(36k + 9)$ with $m_2 = 12(39k^2 + 18k + 2)$ and $m_3 = 39k^2 + 18k + 2$;
- (10⁰) G is a strongly regular graph of order $n = 10(13k - 2)^2$ and degree $r = 3(130k^2 - 40k + 3)$ with $\tau = 90k^2 - 70k + 8$ and $\theta = 15k(6k - 1)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 5k - 1$ and $\lambda_3 = -(60k - 9)$ with $m_2 = 12(130k^2 - 40k + 3)$ and $m_3 = 130k^2 - 40k + 3$;
- ($\bar{10}$ ⁰) G is a strongly regular graph of order $n = 10(13k - 2)^2$ and degree $r = 10(130k^2 - 40k + 3)$ with $\tau = 5(200k^2 - 59k + 4)$ and $\theta = 10(5k - 1)(20k - 3)$, where $k \in \mathbb{N}$.

Its eigenvalues are $\lambda_2 = 60k - 10$ and $\lambda_3 = -5k$ with $m_2 = 130k^2 - 40k + 3$ and $m_3 = 12(130k^2 - 40k + 3)$;

- ($\overline{11}^0$) G is a strongly regular graph of order $n = 10(13k + 2)^2$ and degree $r = 3(130k^2 + 40k + 3)$ with $\tau = 90k^2 + 70k + 8$ and $\theta = 15k(6k + 1)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 60k + 9$ and $\lambda_3 = -(5k + 1)$ with $m_2 = 130k^2 + 40k + 3$ and $m_3 = 12(130k^2 + 40k + 3)$;
- ($\overline{11}^0$) G is a strongly regular graph of order $n = 10(13k + 2)^2$ and degree $r = 10(130k^2 + 40k + 3)$ with $\tau = 5(200k^2 + 59k + 4)$ and $\theta = 10(5k + 1)(20k + 3)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 5k$ and $\lambda_3 = -(60k + 10)$ with $m_2 = 12(130k^2 + 40k + 3)$ and $m_3 = 130k^2 + 40k + 3$;
- ($\overline{12}^0$) G is a strongly regular graph of order $n = 14(13k - 1)^2$ and degree $r = 6(182k^2 - 28k + 1)$ with $\tau = 504k^2 - 119k + 5$ and $\theta = 42k(12k - 1)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 7k - 1$ and $\lambda_3 = -(84k - 6)$ with $m_2 = 12(182k^2 - 28k + 1)$ and $m_3 = 182k^2 - 28k + 1$;
- ($\overline{12}^0$) G is a strongly regular graph of order $n = 14(13k - 1)^2$ and degree $r = 7(182k^2 - 28k + 1)$ with $\tau = 14k(49k - 5)$ and $\theta = 7(7k - 1)(14k - 1)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 84k - 7$ and $\lambda_3 = -7k$ with $m_2 = 182k^2 - 28k + 1$ and $m_3 = 12(182k^2 - 28k + 1)$;
- ($\overline{13}^0$) G is a strongly regular graph of order $n = 14(13k + 1)^2$ and degree $r = 6(182k^2 + 28k + 1)$ with $\tau = 504k^2 + 119k + 5$ and $\theta = 42k(12k + 1)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 84k + 6$ and $\lambda_3 = -(7k + 1)$ with $m_2 = 182k^2 + 28k + 1$ and $m_3 = 12(182k^2 + 28k + 1)$;
- ($\overline{13}^0$) G is a strongly regular graph of order $n = 14(13k + 1)^2$ and degree $r = 7(182k^2 + 28k + 1)$ with $\tau = 14k(49k + 5)$ and $\theta = 7(7k + 1)(14k + 1)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 7k$ and $\lambda_3 = -(84k + 7)$ with $m_2 = 12(182k^2 + 28k + 1)$ and $m_3 = 182k^2 + 28k + 1$;
- ($\overline{14}^0$) G is a strongly regular graph of order $n = 30(13k - 6)^2$ and degree $r = 5(390k^2 - 360k + 83)$ with $\tau = 10(75k^2 - 76k + 19)$ and $\theta = 10(5k - 2)(15k - 7)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 10k - 5$ and $\lambda_3 = -(120k - 55)$ with $m_2 = 12(390k^2 - 360k + 83)$ and $m_3 = 390k^2 - 360k + 83$;
- ($\overline{14}^0$) G is a strongly regular graph of order $n = 30(13k - 6)^2$ and degree $r = 8(390k^2 - 360k + 83)$ with $\tau = 2(960k^2 - 865k + 194)$ and $\theta = 40(2k - 1)(24k - 11)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 120k - 56$ and $\lambda_3 = -(10k - 4)$ with $m_2 = 390k^2 - 360k + 83$ and $m_3 = 12(390k^2 - 360k + 83)$;

- (15⁰) G is a strongly regular graph of order $n = 30(13k + 6)^2$ and degree $r = 5(390k^2 + 360k + 83)$ with $\tau = 10(75k^2 + 76k + 19)$ and $\theta = 10(5k + 2)(15k + 7)$, where $k \geq 0$. Its eigenvalues are $\lambda_2 = 120k + 55$ and $\lambda_3 = -(10k + 5)$ with $m_2 = 390k^2 + 360k + 83$ and $m_3 = 12(390k^2 + 360k + 83)$;
- (15⁰) G is a strongly regular graph of order $n = 30(13k + 6)^2$ and degree $r = 8(390k^2 + 360k + 83)$ with $\tau = 2(960k^2 + 865k + 194)$ and $\theta = 40(2k + 1)(24k + 11)$, where $k \geq 0$. Its eigenvalues are $\lambda_2 = 10k + 4$ and $\lambda_3 = -(120k + 56)$ with $m_2 = 12(390k^2 + 360k + 83)$ and $m_3 = 390k^2 + 360k + 83$;
- (16⁰) G is a strongly regular graph of order $n = 66(13k - 1)^2$ and degree $r = 2(858k^2 - 132k + 5)$ with $\tau = 264k^2 - 143k + 9$ and $\theta = 22k(12k - 1)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 11k - 1$ and $\lambda_3 = -(132k - 10)$ with $m_2 = 12(858k^2 - 132k + 5)$ and $m_3 = 858k^2 - 132k + 5$;
- (16⁰) G is a strongly regular graph of order $n = 66(13k - 1)^2$ and degree $r = 11(858k^2 - 132k + 5)$ with $\tau = 22(11k - 1)(33k - 2)$ and $\theta = 11(11k - 1)(66k - 5)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 132k - 11$ and $\lambda_3 = -11k$ with $m_2 = 858k^2 - 132k + 5$ and $m_3 = 12(858k^2 - 132k + 5)$;
- (17⁰) G is a strongly regular graph of order $n = 66(13k + 1)^2$ and degree $r = 2(858k^2 + 132k + 5)$ with $\tau = 264k^2 + 143k + 9$ and $\theta = 22k(12k + 1)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 132k + 10$ and $\lambda_3 = -(11k + 1)$ with $m_2 = 858k^2 + 132k + 5$ and $m_3 = 12(858k^2 + 132k + 5)$;
- (17⁰) G is a strongly regular graph of order $n = 66(13k + 1)^2$ and degree $r = 11(858k^2 + 132k + 5)$ with $\tau = 22(11k + 1)(33k + 2)$ and $\theta = 11(11k + 1)(66k + 5)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 11k$ and $\lambda_3 = -(132k + 11)$ with $m_2 = 12(858k^2 + 132k + 5)$ and $m_3 = 858k^2 + 132k + 5$.

Proof. First, according to Remark 6 we have $\alpha(\beta - 1) = 12(\alpha - 1)$, from which we find that $\alpha = 2, \beta = 7$ or $\alpha = 3, \beta = 9$ or $\alpha = 4, \beta = 10$ or $\alpha = 6, \beta = 11$ or $\alpha = 12, \beta = 12$. In view of this we obtain the strongly regular graphs represented in Theorem 2 (1⁰), (2⁰), (3⁰), (4⁰), (5⁰). Next, according to Proposition 5 it turns out that G belongs to the class (6⁰) or (7⁰) or (8⁰) or (9⁰) or (10⁰) or (11⁰) or (12⁰) or (13⁰) or (14⁰) or (15⁰) or (16⁰) or (17⁰) if $m_2 = 12m_3$. According to Proposition 6 it turns out that G belongs to the class (6⁰) or (7⁰) or (8⁰) or (9⁰) or (10⁰) or (11⁰) or (12⁰) or (13⁰) or (14⁰) or (15⁰) or (16⁰) or (17⁰) if $m_3 = 12m_2$. \square

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