SCIENTIFIC PUBLICATIONS OF THE STATE UNIVERSITY OF NOVI PAZAR
SER. A: APPL. MATH. INFORM. AND MECH. vol. 16, 1, 2024, 31–37

On the Compact Integral Hypersurfaces of the Multidimensional Homogeneous Darboux System

Vladimir V. Amel'kin, Valentin Yu. Tyshchenko

Dedicated to the memory of Professor Ćemal Dolićanin (1945–2023)

Abstract: The ordinary at $m = 1$ and the completely solvable at $m > 1$ the real homogeneous Darboux systems

$$dx_i = \sum_{j=1}^{m} \Big(L_{ij}(x) + x_i P_j(x)\Big) dt_j, i = \overline{1,n},$$

where $n > 1,\ m < n,\ L_{ij}(x) = \sum_{k=1}^{n} a_{ijk} x_k,\ a_{ijk} \in \mathbb{R},\ k = \overline{1,n},\ j = \overline{1,m},\ i = \overline{1,n},\ P_j(x),\ j = \overline{1,m}$, are twice smooth homogeneous functions of order $\rho \geqslant 1$, and the rank of the matrix corresponding to the right-hand sides of the Darboux systems is equal to m almost everywhere on $\mathbb{R}^n$, are considered. The concept of a regular integral hypersurface of these systems is introduced. It is proved that the systems under consideration cannot have more than one isolated compact regular integral hypersurface. An example of a system with an isolated compact regular integral hypersurface is given.

Keywords: system of ordinary differential equations, completely solvable system of exact differential equations, hypersurface.

1 Introduction

In 1878 Gaston Darboux published the work [2] in which the properties of first-order differential equation

$$\frac{dy}{dx} = \frac{L(x,y) + yN(x,y)}{M(x,y) + xN(x,y)}, \tag{1}$$

where $L(x,y),\ M(x,y),\ N(x,y)$ are polynomials, were considered. Subsequently in the articles [9, 10] and the monographs [1, 8] the qualitative characteristics of special cases of the

Manuscript received February 22, 2024; accepted March 4, 2024
Vladimir V. Amel'kin is with Belarusian State University, Minsk, Belarus;
Valentin Yu. Tyshchenko is with Grodno State University, Grodno, Belarus
https://doi.org/10.46793/SPSUNP2401.031A

equation (1) were considered. The n–dimensional analogue of the equation (1)

$$\frac{dw_i}{dz} = L_i(z,w) + w_i M(z,w),\ i = \overline{1,n}, \tag{2}$$

where $n > 1,\ w = (w_1, \ldots, w_n),\ L_i(z,w) = \sum\limits_{k=1}^{n} a_{ik}(z) w_k,\ a_{ik}(z)$ are holomorphic functions, $k = \overline{1,n},\ i = \overline{1,n},\ M(z,w)$ is homogeneous function with respect to w of order ρ with holomorphic coefficients, was considered in the work [7]. For the system (2) the solutions and the integrals were studied in the complex case, and the limit cycles were studied in the autonomous real case. In the same work the system (2) was called **Darboux system**.

In this work we will consider the compact integral hypersurfaces of the ordinary at $m = 1$ and the completely solvable [4, p. 21] at $m > 1$ the real differential system

$$dx_i = \sum_{j=1}^{m} \Big(L_{ij}(x) + x_i P_j(x) \Big) dt_j,\ i = \overline{1,n}, \tag{3}$$

where $n > 1,\ m < n,\ L_{ij}(x) = \sum\limits_{k=1}^{n} a_{ijk} x_k,\ a_{ijk} \in \mathbb{R},\ k = \overline{1,n},\ j = \overline{1,m},\ i = \overline{1,n},\ P_j(x), j = \overline{1,m}$, are twice smooth homogeneous functions of order $\rho \geqslant 1$, and the rank of the matrix corresponding to the right-hand sides of Darboux systems is equal to m almost everywhere on $\mathbb{R}^n$. We will call it **homogeneous Darboux system**.

Further in particular we will use the following statements.

Consider the real ordinary differential system

$$\frac{dx_i}{dt} = f_i(x),\ i = \overline{1,n}, \tag{4}$$

where $n > 1,\ f_i(x),\ i = \overline{1,n}$, are smooth functions.

Proposition 1 [6, p. 345]. *Let the domain $G \subset \mathbb{R}^n$ have the homotopy group $\pi_{n-1}(G)$ of rank r and there exist the continuously differentiable function $\varphi : G \to \mathbb{R}$ such that the vector field $\varphi(x)(f_1(x), \ldots, f_n(x))$ has a constant sign divergence on the domain G. Then the system* (4) *cannot have more than r compact integral hypersurfaces on the domain G.*

Proposition 2 [6, p. 349]. *Let the domain $G \subset \mathbb{R}^n$ have the homotopy group $\pi_{n-1}(G)$ of rank r and there exist the continuously differentiable function $\varphi : G \to \mathbb{R}$ such that the vector field $\varphi(x)(f_1(x), \ldots, f_n(x))$ has identically equal to zero divergence on the domain G. Then the system* (4) *cannot have more than r isolated compact regular integral hypersurfaces on the domain G.*

Proposition 3 [6, p. 324]. *Let the real completely solvable differential system*

$$dx_i = \sum_{j=1}^{m} f_{ij}(x) dt_j,\ i = \overline{1,n},$$

where $n > 1,\ m > 1,\ m < n,\ f_{ij}(x),\ i = \overline{1,n},\ j = \overline{1,m}$, are smooth functions, has a integral manifold. Then each of the ordinary differential systems

$$\frac{dx_i}{dt} = f_{ij}(x),\ i = \overline{1,n},$$

has the same integral manifold, $j = \overline{1,m}$.

2 Results

First we consider the ordinary homogeneous Darboux system

$$\frac{dx_i}{dt} = L_i(x) + x_i P(x), \ i = \overline{1,n}, \tag{5}$$

where $n > 1$, $L_i(x) = \sum_{k=1}^{n} a_{ik} x_k$, $a_{ik} \in \mathbb{R}$, $k = \overline{1,n}$, $i = \overline{1,n}$, $P(x)$ is twice smooth homogeneous function of order $\rho \geqslant 1$.

Definition 1. *We will call a compact integral hypersurface of the system* (5) **regular** *if this hypersurface is locally attractive or locally repulsive in each of the two half-neighborhoods it defines and the closure of a trajectory of each point of this hypersurface coincides with the entire hypersurface.*

Let us choose the auxiliary linear function

$$H(x) = \sum_{l=1}^{n} A_l x_l, \tag{6}$$

where $A_l \in \mathbb{C}$, $l = \overline{1,n}$. Its derivative due to the system (5) is equal to

$$\left.\frac{dH(x)}{dt}\right|_{(5)} = H(x)P(x) + \sum_{l=1}^{n} A_l L_l(x). \tag{7}$$

We will look for the numbers A_l, $l = \overline{1,n}$, such that the identity

$$\sum_{l=1}^{n} A_l L_l(x) \equiv \lambda H(x), \ \lambda \in \mathbb{C} \tag{8}$$

holds. The identity (8) is equivalent to the system of linear homogeneous equations

$$\sum_{l=1}^{n} a_{li} A_l - \lambda A_l = 0, i = \overline{1,n}. \tag{9}$$

The system (9) has non-zero solutions only for values of λ satisfying its characteristic equation

$$\begin{vmatrix} a_{11} - \lambda & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{21} - \lambda & \cdots & a_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} - \lambda \end{vmatrix} = 0. \tag{10}$$

We will call the characteristic equation (10) **characteristic equation of the system** (5).

If the characteristic equation (10) has at least one real characteristic root $\lambda \in \mathbb{R}$ then based on it and the system (9) we construct the real linear function (6). Further by virtue of the relations (7) and (8) we obtain that the derivative due to the system (5) is equal to

$$\left.\frac{dH(x)}{dt}\right|_{(5)} = H(x)(\lambda + P(x)). \tag{11}$$

There is the relation

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i}(L_i(x) + x_i P(x)) \equiv \sum_{i=1}^{n} a_{ii} + (n+\rho)P(x), \tag{12}$$

Then we have the identity

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\Big(H(x)^{-(n+\rho)}(L_i(x) + x_i P(x))\Big) \equiv \left(\sum_{i=1}^{n} a_{ii} - (n+\rho)\lambda\right) H(x)^{-(n+\rho)} \tag{13}$$

(based on the relations (11) and (12)). Now using the identity (13), based on Definition 1 and Propositions 1 and 2, we obtain the following statements.

Theorem 1. *Suppose that the characteristic equation of the system* (5) *has at least one real root. Then this system does not have isolated compact regular integral hypersurfaces.*

Corollary 1. *The system* (5) *does not have isolated compact regular integral hypersurfaces for odd n.*

Let us now consider the case when the characteristic equation (10) of the system (5) has no real roots. Let $\lambda \in \mathbb{C}$, $Im\, \lambda \neq 0$, be the root of the characteristic equation (10). We construct the linear function (6) of real variables with complex coefficients based on this root and the system (9). By virtue of the formulas (7) and (8) we have the relations

$$\begin{aligned} \left.\frac{d(Re\, H(x))}{dt}\right|_{(5)} &= Re\, \lambda\ (Re\, H(x)) - Im\, \lambda\ (Im\, H(x)) + (Re\, H(x))\ P(x), \\ \left.\frac{d(Im\, H(x))}{dt}\right|_{(5)} &= Re\, \lambda\ (Im\, H(x)) + Im\, \lambda\ (Re\, H(x)) + (Im\, H(x))\ P(x). \end{aligned} \tag{14}$$

Based on the last two relations we obtain the expression

$$\left.\frac{d\ (Re^2 H(x) + Im^2 H(x))}{dt}\right|_{(5)} = 2(Re^2 H(x) + Im^2 H(x))(Re\, \lambda + P(x)). \tag{15}$$

Now taking into account relations (12), (14) and (15) we obtain the identity

$$\begin{aligned} &\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\Big((Re^2 H(x) + Im^2 H(x))^{-(n+\rho)}(L_i(x) + x_i P(x))\Big) \equiv \\ &\equiv \left(\sum_{i=1}^{n} a_{ii} - (n+\rho) Re\, \lambda\right)(Re^2 H(x) + Im^2 H(x))^{-(n+\rho)}. \end{aligned} \tag{16}$$

Let us show that the system (5) has at most one isolated compact regular integral hypersurface in the domain $G = \mathbb{R}^n \setminus \{Re^2 H(x) + Im^2 H(x) = 0\}$. In fact we suppose that the system (5) has entirely located in the domain G two various compact regular integral hypersurfaces Φ_1 and Φ_2. Each of these hypersurfaces is located in a subdomain of the domain G that is not contractible to a point by Propositions 1 and 2. In addition the given hypersurfaces Φ_1 and Φ_2 define the elements σ_1 and σ_2 in the $(n-1)$-dimensional homology group $H_{n-1}(G)$ of the domain G. The Abelian group $H_{n-1}(G) = \mathbb{Z}$ has at most one linearly independent element. Therefore we come to the conclusion that there is such a non-trivial linear combination with integer coefficients: $m_1\sigma_1 + m_2\sigma_2 = O \in H_{n-1}(G), (m_1, m_2) \neq (0,0)$. This means that the composite compact regular integral hypersurface $\Phi = m_1\sigma_1 + m_2\sigma_2$, which is the $(n-1)$-dimensional cycle, is homologous to zero (i.e., it bounds some n-dimensional domain $V \subset G$). Now based on the identity (16) by virtue of the general Stokes formula [3, p. 472] (for $\sum\limits_{i=1}^{n} a_{ii} - (n+\rho)Re\ \lambda) \neq 0$) and Ostrogradsky formula [11, pp. 324, 588–593] (for $\sum\limits_{i=1}^{n} a_{ii} - (n+\rho)Re\ \lambda) = 0$) we obtain a contradiction. Thus we obtain the following statements taking into account Theorem 1.

Theorem 2. *Suppose that the characteristic equation of the system* (5) *has no real roots. Then this system cannot have more than one isolated compact regular integral hypersurface.*

Corollary 2. *The system* (5) *cannot have more than one isolated compact regular integral hypersurface.*

Example [8, p. 58]. The system

$$\frac{dx}{dt} = x + y - x(10x^2 - y^2),\ \frac{dy}{dt} = -x + y - y(10x^2 - y^2),$$

has the unique limit cycle $29x^2 - 22xy + 7y^2 = 4$.

Let us now consider the completely solvable homogeneous Darboux system (3).

Definition 2. *We will call a compact integral hypersurface of system* (3) **regular** *if it is regular for each vector field* $(L_{1j}(x) + x_1P_j(x), \dots, (L_{nj}(x) + x_nP_j(x)),\ j = \overline{1,m}$.

We will compose the auxiliary linear function (6) (as in the case of the ordinary differential system (5)). Its differential due to the system (3) is equal to

$$dH(x)\Big|_{(3)} = H(x)\sum_{j=1}^{m} P_j(x)dt_j + \sum_{j=1}^{m}\sum_{l=1}^{n} A_l L_{lj}(x)dt_j. \tag{17}$$

We will look for the numbers $A_l, l = \overline{1,n}$, such that the identities

$$\sum_{l=1}^{n} A_l L_l j(x) \equiv \lambda_j H(x),\ \lambda_j \in \mathbb{C},\ j = \overline{1,m} \tag{18}$$

are satisfied. The identities (18) are equivalent to the system of linear homogeneous equations

$$\sum_{l=1}^{n} a_{lij}A_l - \lambda_j A_l = 0,\ i = \overline{1,n},\ j = \overline{1,m}. \tag{19}$$

Since the system (3) is completely solvable, the matrices $a_j = ||a_{lij}||,\ j = \overline{1,m}$, are commutable. Therefore the collection of these permutational matrices has the common non-zero eigenvector $(A_1,\dots,A_n)$ by Lemma 1 [5, p.230].

Let the vector $(A_1,\dots,A_n)$ be real. We compose the auxiliary function (6) based on it. We obtain the relation

$$dH(x)\Big|_{(3)} = H(x)\sum_{j=1}^{m}(\lambda_j + P_j(x))dt_j \tag{20}$$

taking into account the formulas (17) – (19). Since the relations

$$\sum_{i=1}^{n}\frac{\partial}{\partial x_i}(L_ij(x) + x_iP_j(x)) \equiv \sum_{i=1}^{n} a_{iij} + (n+\rho)P_j(x),\ j = \overline{1,m}, \tag{21}$$

are valid then based on the formula (20) we have the identities

$$\begin{gathered}\sum_{i=1}^{n}\tfrac{\partial}{\partial x_i}\Big(H(x)^{-(n+\rho)}(L_ij(x) + x_iP_j(x))\Big) \equiv \\ \equiv \left(\sum_{i=1}^{n} a_{iij} - (n+\rho)\lambda_j\right)H(x)^{-(n+\rho)},\ j = \overline{1,m}.\end{gathered} \tag{22}$$

Now using the identities (21) and (22) based on Definition 2 and Propositions 1 – 3, we obtain the statement.

Theorem 3. *Suppose that the permutational matrices $a_j,\ j = \overline{1,m}$, have a real common eigenvector. Then system* (3) *does not have isolated compact regular integral hypersurfaces.*

Let us now consider the case when the permutational matrices $a_j, j = \overline{1,m}$, do not have real common eigenvectors. Then they have the common complex eigenvector $(A_1,\dots,A_n)$. Let's create the auxiliary function (6) based on it. Next we construct the linear function (6) of real variables with complex coefficients based on this eigenvector and the system (9). By virtue of the relations (17) – (19) we will have that

$$\begin{aligned} d(Re\ H(x))\Big|_{(3)} &= (Re\ H(x))\sum_{j=1}^{m}(Re\ \lambda_j + P_j(x))dt_j - (Im\ H(x))\sum_{j=1}^{m}(Im\ \lambda_j)dt_j, \\ d(Im\ H(x))\Big|_{(3)} &= (Re\ H(x))\sum_{j=1}^{m}(Im\ \lambda_j)dt_j + (Im\ H(x))\sum_{j=1}^{m}(Re\ \lambda_j + P_j(x))dt_j. \end{aligned} \tag{23}$$

Based on the last two relations we have the expression

$$d(Re^2H(x) + Im^2H(x))\Big|_{(3)} = 2(Re^2H(x) + Im^2H(x))\sum_{j=1}^{m}(Re\ \lambda_j + P_j(x))dt_j. \tag{24}$$

Now taking into account the relations (21), (23) and (24) we obtain the identities

$$\begin{gathered}\sum_{i=1}^{n}\tfrac{\partial}{\partial x_i}\Big((Re^2H(x) + Im^2H(x))^{-(n+\rho)}(L_(ij)(x) + x_iP_j(x))\Big) \equiv \\ \equiv \left(\sum_{i=1}^{n} a_{iij} - (n+\rho)Re\ \lambda_j\right)(Re^2H(x) + Im^2H(x))^{-(n+\rho)},\ j = \overline{1,m}.\end{gathered} \tag{25}$$

And then we obtain such statements based on the identity (25) by virtue of Proposition 3 and Theorem 2.

Theorem 4. *Suppose that the permutational matrices* $a_j,\ j=\overline{1,m}$, *do not have real common eigenvectors. Then the system* (3) *cannot have more than one isolated compact regular integral hypersurface.*

Corollary 3. *The system* (3) *cannot have more than one isolated compact regular integral hypersurface.*

References

[1] V. V. AMEL'KIN, N. A. LUKASHEVICH, A. P. SADOVSKY, *Nonlinear Oscillations in Second Order Systems*, Publishing house of BSU, Minsk, 1982.

[2] G. DARBOUX, *Mèmoire sur les èquations diffèrentielles algèbriques du premier ordre et du premier degrè*, Bull. des sci. Vol. 2 (1878), 60–96.

[3] B. A. DUBROVIN, S. P. NOVIKOV, A. T. FOMENKO, *Modern Geometry: Methods and Applications*, Nauka, Moscow, 1986.

[4] I. V. GAISHUN, *Completely Solvable Multidimensional Differential Equations*, URSS, Moscow, 2004.

[5] F. R. GANTMAKHER, *Matrix Theory*, Nauka, Moscow, 1988.

[6] V. N. GORBUZOV, *Integrals of Differential Systems*, GrSU, Grodno, 2006.

[7] V. N. GORBUZOV, P. B. PAVLYUCHIK, *Solutions, integrals and limit cycles of the Darboux system of n–th order*, Differential Equations and Control Processes. 2 (2002), 26–46.

[8] V. N. GORBUZOV, A. A. SAMODUROV, *Darboux Equation and its Analogues*, GrSU, Grodno, 1985.

[9] N. A. LUKASHEVICH, *Integral curves of the Darboux equation*, Differential Equations. Vol. 2, 5 (1966), 628–633.

[10] M. M. MAKHANYOK, *Limit cycles of the Darboux system*, Proceedings of the National Academy of Sciences of Belarus. Physics and Mathematics series. 1 (1966), 6–11.

[11] G. E. SHILOV, *Mathematical Analysis. Functions of Several Variables.*, Nauka, Moscow, 1972.