

On Some Vertex Degree Based Topological Indices of Graphs and Corresponding Polynomials

Mert Sinan Öz, Emir Zogić

Abstract: Let G be a simple graph with n vertices and m edges. To each graph G we can associate the following matrix $A_{\mathcal{F}} = [a_{ij}]_{n \times n}$ with

$$a_{ij} = \begin{cases} \mathcal{F}(d_i, d_j), & i \sim j \\ 0, & \text{otherwise,} \end{cases}$$

where $\mathcal{F}$ is an appropriately chosen function with the property $\mathcal{F}(x, y) = \mathcal{F}(y, x)$. Let $P_{TI(G)}(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + a_3x^{n-3} + \dots + a_n$ be the characteristic polynomial of the matrix $A_{\mathcal{F}}$. In this paper, we consider the properties of the coefficients a_1 and a_2 . Furthermore, we present some bounds of the $\sum_{i \sim j} \mathcal{F}(d_i, d_j)^2$ by means of the eigenvalues of the matrix $A_{\mathcal{F}}$. Using these general results, we obtained some new bounds for the general Randić index R_{-1} of bipartite graphs.

Keywords: Topological indices, the general Randić index, coefficients of the characteristic polynomial

1 Introduction

Let $G = (V, E)$ be a simple connected graph with n vertices and m edges with vertex degrees $d_1 \geq d_2 \geq \dots \geq d_n > 0$. By $i \sim j$ we denote that i and j are the adjacent vertices.

In chemical and spectral graph theory, there are many graph invariants, so-called topological indices which are of interest for research in mathematics and chemistry. One can see that

$$TI = TI(G) = \sum_{i \sim j} \mathcal{F}(d_i, d_j)$$

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is a representation of topological indices based on vertex degrees, where $\mathcal{F}$ is an appropriately chosen function with the property $\mathcal{F}(x, y) = \mathcal{F}(y, x)$ [11, 12].

To make a connection between TI and matrices, let us recall the definition of the well-known adjacency matrix $A = [a_{ij}]_{n \times n}$ as

$$a_{ij} = \begin{cases} 1, & i \sim j \\ 0, & \text{otherwise.} \end{cases}$$

From [3] we can make a certain generalization by introducing the following matrix $A_{\mathcal{F}} = [a_{ij}]_{n \times n}$:

$$a_{ij} = \begin{cases} \mathcal{F}(d_i, d_j), & i \sim j \\ 0, & \text{otherwise,} \end{cases}$$

and this matrix can be associated to each topological index TI .

In [15] lower and upper bounds were obtained for the eigenvalues of the Randić vertex-degree-based adjacency matrix RA . In this paper, as a continuation of [15], we present some bounds of the $\sum_{i \sim j} \mathcal{F}(d_i, d_j)^2$ using the eigenvalues of $A_{\mathcal{F}}$. We also obtain some bounds for the general Randić index of bipartite graphs.

2 Main Results

A subgraph of G , each component of which is an edge or a cycle, is called an elementary subgraph of G [1].

Lemma 2.1 *Let $P_{TI(G)}(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + a_3x^{n-3} + \dots + a_n$ be the characteristic polynomial of the matrix $A_{\mathcal{F}}$. Then*

$$a_1 = 0, a_2 = - \sum_{i \sim j} \mathcal{F}(d_i, d_j)^2. \quad (1)$$

Proof. Using Theorem 3.8 in [1], since an elementary subgraph with 1 vertex in G is not possible, it is obtained that $a_1 = 0$. Moreover, since elementary subgraphs with 2 vertices correspond to each edge itself, a_2 is equal to -1 times the sum of the products of $\mathcal{F}(d_i, d_j) \cdot \mathcal{F}(d_j, d_i)$ for each edge with end vertices i and j , which is

$$a_2 = - \sum_{i \sim j} \mathcal{F}(d_i, d_j) \cdot \mathcal{F}(d_j, d_i) = - \sum_{i \sim j} \mathcal{F}(d_j, d_i)^2.$$

□

Let $\mathcal{P}_n(a_1, a_2)$ be a class of real polynomials of the form $x^n + a_1x^{n-1} + a_2x^{n-2} + a_3x^{n-3} + \dots + a_n$, where a_1 and a_2 are fixed real numbers.

It is clear that $P_{TI(G)}(x)$ belongs to class $\mathcal{P}_n \left(0, - \sum_{i \sim j} \mathcal{F}(d_i, d_j)^2 \right)$ by Lemma 2.1.

Let $f_1 \geq f_2 \geq \dots \geq f_n$ be eigenvalues of matrix $A_{\mathcal{F}}$. In [3], the following equations were presented:

$$\text{tr}(A_{\mathcal{F}}) = f_1 + f_2 + \dots + f_n = 0, \quad (2)$$

$$\text{tr}(A_{\mathcal{F}}^2) = \sum_{i=1}^n f_i^2 = 2 \sum_{i \sim j} \mathcal{F}(d_i, d_j)^2. \quad (3)$$

Since the class $P_{TI(G)}(x)$ belongs to the class $\mathcal{P}_n \left(0, -\sum_{i \sim j} \mathcal{F}(d_i, d_j)^2 \right)$, we can apply the method presented in [9] to $P_{TI(G)}(x)$ and obtain the following inequalities:

$$\bar{x} + \frac{1}{n} \sqrt{\frac{\Delta}{n-1}} \leq f_1 \leq \bar{x} + \frac{1}{n} \sqrt{(n-1)\Delta}, \quad (4)$$

$$\bar{x} - \frac{1}{n} \sqrt{\frac{(i-1)\Delta}{n-i+1}} \leq f_i \leq \bar{x} + \frac{1}{n} \sqrt{\frac{(n-1)\Delta}{i}}, \quad \text{where } i = 2, 3, \dots, n-1, \quad (5)$$

$$\bar{x} - \frac{1}{n} \sqrt{(n-1)\Delta} \leq f_n \leq \bar{x} - \frac{1}{n} \sqrt{\frac{\Delta}{n-1}}, \quad (6)$$

where

$$\bar{x} = \frac{1}{n} \sum_{i=1}^n f_i \text{ and } \Delta = n \sum_{i=1}^n f_i^2 - \left(\sum_{i=1}^n f_i \right)^2.$$

It is obtained by Equations (2) and (3) that

$$\bar{x} = \frac{1}{n} \sum_{i=1}^n f_i = 0 \text{ and } \Delta = n \sum_{i=1}^n f_i^2 - \left(\sum_{i=1}^n f_i \right)^2 = 2n \sum_{i \sim j} \mathcal{F}(d_i, d_j)^2.$$

Theorem 2.1 *Let G be a connected graph with $n \geq 2$ vertices with the general type adjacency matrix $A_{\mathcal{F}}$, the general type characteristic polynomial $P_{TI(G)}(x)$ and the eigenvalues $f_1 \geq f_2 \geq \dots \geq f_n$. Then, the following inequalities hold:*

$$\frac{n}{2(n-1)} f_1^2 \leq \sum_{i \sim j} \mathcal{F}(d_i, d_j)^2 \leq \frac{n(n-1)}{2} f_1^2,$$

$$\frac{n}{2(n-1)} f_n^2 \leq \sum_{i \sim j} \mathcal{F}(d_i, d_j)^2 \leq \frac{n(n-1)}{2} f_n^2.$$

Proof. Applying inequality (4) to $P_{TI(G)}(x)$, i.e. to

$$x^n + a_1x^{n-1} + a_2x^{n-2} + a_3x^{n-3} + \cdots + a_n,$$

the following inequality is obtained:

$$\frac{1}{n} \sqrt{\frac{2n \sum_{i \sim j} \mathcal{F}(d_i, d_j)^2}{n-1}} \leq f_1 \leq \frac{1}{n} \sqrt{(n-1) 2n \sum_{i \sim j} \mathcal{F}(d_i, d_j)^2}.$$

As a result, the following result follows straightforwardly:

$$\frac{n}{2(n-1)} f_1^2 \leq \sum_{i \sim j} \mathcal{F}(d_i, d_j)^2 \leq \frac{n(n-1)}{2} f_1^2.$$

Similarly, applying inequality (6) to $P_{TI(G)}(x)$ reveals the following inequality:

$$-\frac{1}{n} \sqrt{(n-1) 2n \sum_{i \sim j} \mathcal{F}(d_i, d_j)^2} \leq f_n \leq -\frac{1}{n} \sqrt{\frac{2n \sum_{i \sim j} \mathcal{F}(d_i, d_j)^2}{n-1}}.$$

The result is obtained as follows:

$$\frac{1}{n} \sqrt{(n-1) 2n \sum_{i \sim j} \mathcal{F}(d_i, d_j)^2} \geq -f_n \geq \frac{1}{n} \sqrt{\frac{2n \sum_{i \sim j} \mathcal{F}(d_i, d_j)^2}{n-1}},$$

$$\sqrt{\frac{2(n-1) \sum_{i \sim j} \mathcal{F}(d_i, d_j)^2}{n}} \geq -f_n \geq \sqrt{\frac{2 \sum_{i \sim j} \mathcal{F}(d_i, d_j)^2}{n(n-1)}},$$

$$\frac{2(n-1) \sum_{i \sim j} \mathcal{F}(d_i, d_j)^2}{n} \geq f_n^2 \geq \frac{2 \sum_{i \sim j} \mathcal{F}(d_i, d_j)^2}{n(n-1)},$$

$$\frac{n}{2(n-1)} f_n^2 \leq \sum_{i \sim j} \mathcal{F}(d_i, d_j)^2 \leq \frac{n(n-1)}{2} f_n^2.$$

□

Theorem 2.2 Let G be a connected graph with $n \geq 2$ vertices with the general type adjacency matrix $A_{\mathcal{F}}$, the general type characteristic polynomial $P_{TI(G)}(x)$ and the eigenvalues $f_1 \geq f_2 \geq \cdots \geq f_n$.

For each f_i satisfying the condition $0 \leq f_i$ except f_1 and f_n , the following inequality holds:

$$\frac{n \cdot i}{2(n-i)} f_i^2 \leq \sum_{i \sim j} \mathcal{F}(d_i, d_j)^2. \quad (7)$$

For each f_i satisfying the condition $0 \leq \sqrt{\frac{2(i-1)}{n(n-i+1)} \sum_{i \sim j} \mathcal{F}(d_i, d_j)^2} \leq f_i$ except f_1 and f_n , the following inequality holds:

$$\sum_{i \sim j} \mathcal{F}(d_i, d_j)^2 \leq \frac{n(n-i+1)}{2(i-1)} f_i^2. \quad (8)$$

For each f_i satisfying the condition $0 \leq f_i \leq \sqrt{\frac{2(i-1)}{n(n-i+1)} \sum_{i \sim j} \mathcal{F}(d_i, d_j)^2}$ except f_1 and f_n , the following inequality holds:

$$\frac{n(n-i+1)}{2(i-1)} f_i^2 \leq \sum_{i \sim j} \mathcal{F}(d_i, d_j)^2. \quad (9)$$

For each f_i satisfying the condition $f_i \leq 0$ except f_1 and f_n , the following inequality holds:

$$\sum_{i \sim j} \mathcal{F}(d_i, d_j)^2 \geq \frac{n(n-i+1)}{2(i-1)} f_i^2. \quad (10)$$

For each f_i satisfying the condition $-\sqrt{\frac{2(n-1)}{n \cdot i} \sum_{i \sim j} \mathcal{F}(d_i, d_j)^2} \leq f_i \leq 0$ except f_1 and f_n , the following inequality holds:

$$\sum_{i \sim j} \mathcal{F}(d_i, d_j)^2 \geq \frac{n \cdot i}{2(n-1)} f_i^2. \quad (11)$$

For each f_i satisfying the condition $f_i \leq -\sqrt{\frac{2(n-1)}{n \cdot i} \sum_{i \sim j} \mathcal{F}(d_i, d_j)^2} \leq 0$ except f_1 and f_n , the following inequality holds:

$$\frac{n \cdot i}{2(n-1)} f_i^2 \geq \sum_{i \sim j} \mathcal{F}(d_i, d_j)^2. \quad (12)$$

Proof. Applying inequality (5) to $P_{TI(G)}(x)$

$$x^n + a_1 x^{n-1} + a_2 x^{n-2} + a_3 x^{n-3} + \cdots + a_n,$$

the following inequality is obtained:

$$-\frac{1}{n}\sqrt{\frac{2n(i-1)}{n-i+1}\sum_{i\sim j}\mathcal{F}(d_i,d_j)^2}\leq f_i\leq\frac{1}{n}\sqrt{\frac{2n(n-1)}{i}\sum_{i\sim j}\mathcal{F}(d_i,d_j)^2},$$

where $i = 2, 3, \dots, n-1$. In simpler form

$$-\sqrt{\frac{2(i-1)}{n(n-i+1)}\sum_{i\sim j}\mathcal{F}(d_i,d_j)^2}\leq f_i\leq\sqrt{\frac{2(n-1)}{n\cdot i}\sum_{i\sim j}\mathcal{F}(d_i,d_j)^2}, \quad (13)$$

where $i = 2, 3, \dots, n-1$.

Let us continue by analyzing this inequality into two fundamental cases with subcases. Let us begin with the first case of f_i .

Case I: Let us assume that $0 \leq f_i$.

Case I.I: Since $0 \leq f_i$ and $f_i \leq \sqrt{\frac{2(n-1)}{n\cdot i}\sum_{i\sim j}\mathcal{F}(d_i,d_j)^2}$ in equation (13), it is obtained that

$$\frac{n\cdot i}{2(n-i)}f_i^2\leq\sum_{i\sim j}\mathcal{F}(d_i,d_j)^2.$$

Case I.II: Since $0 \leq f_i$ and $-\sqrt{\frac{2(i-1)}{n(n-i+1)}\sum_{i\sim j}\mathcal{F}(d_i,d_j)^2}\leq f_i$ in equation (13), it is needed to consider two more conditions to get some bounds as follows:
The first condition is

$$\left|-\sqrt{\frac{2(i-1)}{n(n-i+1)}\sum_{i\sim j}\mathcal{F}(d_i,d_j)^2}\right|=\sqrt{\frac{2(i-1)}{n(n-i+1)}\sum_{i\sim j}\mathcal{F}(d_i,d_j)^2}\leq|f|=f_i,$$

and the second condition is

$$\left|-\sqrt{\frac{2(i-1)}{n(n-i+1)}\sum_{i\sim j}\mathcal{F}(d_i,d_j)^2}\right|=\sqrt{\frac{2(i-1)}{n(n-i+1)}\sum_{i\sim j}\mathcal{F}(d_i,d_j)^2}\geq|f|=f_i,$$

If the first condition applies, which is $0 \leq \sqrt{\frac{2(i-1)}{n(n-i+1)}\sum_{i\sim j}\mathcal{F}(d_i,d_j)^2}\leq f_i$ then it is obtained that

$$\sum_{i\sim j}\mathcal{F}(d_i,d_j)^2\leq\frac{n(n-i+1)}{2(i-1)}f_i^2$$

If the second condition applies, which is $0 \leq f_i \leq \sqrt{\frac{2(i-1)}{n(n-i+1)} \sum_{i \sim j} \mathcal{F}(d_i, d_j)^2}$ then it is achieved that

$$\frac{n(n-i+1)}{2(i-1)} f_i^2 \leq \sum_{i \sim j} \mathcal{F}(d_i, d_j)^2$$

Now, let us focus on the second case of f_i .

Case II: Let us assume that $f_i \leq 0$.

Case II.I: Since $f_i \leq 0$ and $-\sqrt{\frac{2(i-1)}{n(n-i+1)} \sum_{i \sim j} \mathcal{F}(d_i, d_j)^2} \leq f_i$ in the equation (13), it is obtained that

$$\sum_{i \sim j} \mathcal{F}(d_i, d_j)^2 \geq \frac{n(n-i+1)}{2(i-1)} f_i^2.$$

Case II.II: Since $f_i \leq 0$ and $f_i \leq \sqrt{\frac{2(n-1)}{n \cdot i} \sum_{i \sim j} \mathcal{F}(d_i, d_j)^2}$ in equation (13), it is needed to consider two more conditions to get some more bounds according to cases for ρ_i as follows:

The first condition is

$$|f_i| = -f_i \leq \left| \sqrt{\frac{2(n-1)}{n \cdot i} \sum_{i \sim j} \mathcal{F}(d_i, d_j)^2} \right| = \sqrt{\frac{2(n-1)}{n \cdot i} \sum_{i \sim j} \mathcal{F}(d_i, d_j)^2},$$

and the second condition is

$$|f_i| = -f_i \geq \left| \sqrt{\frac{2(n-1)}{n \cdot i} \sum_{i \sim j} \mathcal{F}(d_i, d_j)^2} \right| = \sqrt{\frac{2(n-1)}{n \cdot i} \sum_{i \sim j} \mathcal{F}(d_i, d_j)^2},$$

If the first condition applies, which is $-\sqrt{\frac{2(n-1)}{n \cdot i} \sum_{i \sim j} \mathcal{F}(d_i, d_j)^2} \leq f_i \leq 0$ then it is obtained that

$$\sum_{i \sim j} \mathcal{F}(d_i, d_j)^2 \geq \frac{n \cdot i}{2(n-1)} f_i^2.$$

If the second condition applies, which is $f_i \leq -\sqrt{\frac{2(n-1)}{n \cdot i} \sum_{i \sim j} \mathcal{F}(d_i, d_j)^2} \leq 0$ then it is achieved that

$$\frac{n \cdot i}{2(n-1)} f_i^2 \geq \sum_{i \sim j} \mathcal{F}(d_i, d_j)^2.$$

□

3 Some new bounds for the general Randić index R_{-1}

For $\mathcal{F}(d_i, d_j) = \frac{1}{\sqrt{d_i d_j}}$ we have that $TI = R$ is the classical Randić index. By Lemma 2.1 it is $a_2 = -R_{-1}$ for the corresponding characteristic polynomial of matrix $A_{\mathcal{F}}$, where R_{-1} is another version of the Randić index, in literature so-called the general Randić index and it's defined as $R_{-1} = \sum_{i \sim j} \frac{1}{d_i d_j}$ [10].

Lemma 3.1 *Let G be a connected graph with $n \geq 2$ vertices with the Randić polynomial $P_{R(G)}(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + a_3 x^{n-3} + \dots + a_n$. Then, the following holds:*

$$a_1 = 0, a_2 = -R_{-1}(G). \quad (14)$$

Proof. The proof follows from the Lemma 2.1. $\square$

Let $G_{r,s}$ denote a connected bipartite graph with $r + s$ vertices. In the case of $G \cong G_{r,s}$ for real numbers $a_1, a_2, a_3, \dots, a_{n-2}$ it can be deduced that

$$P_{R(G_{r,s})}(x) = (x^2 - 1) \prod_{i=2}^{n-1} (x - \rho_i) = (x^2 - 1)(x^{n-2} + a_1 x^{n-3} + a_2 x^{n-4} + a_3 x^{n-5} + \dots + a_{n-2})$$

corresponds to the general form of the Randić polynomial of $G_{r,s}$ with eigenvalues $\rho_1 \geq \rho_2 \geq \dots \geq \rho_n$, where $\rho_1 = 1, \rho_n = -1$.

Lemma 3.2 *Let $G_{r,s}$ be a connected bipartite graph with $r + s = n \geq 4$ vertices with the Randić polynomial $P_{R(G_{r,s})}(x)$. Then, the following holds:*

$$\text{For each odd } i, \text{ it is obtained that } a_i = 0, a_2 = 1 - R_{-1}(G_{r,s}). \quad (15)$$

Proof. The proof follows from the property that no bipartite graph has an odd cycle. $\square$

It is worth noting that $P_{R(G)}(x)$ and the factor $(x^{n-2} + a_1 x^{n-3} + a_2 x^{n-4} + a_3 x^{n-5} + \dots + a_{n-2})$ of $P_{R(G_{r,s})}(x)$ belong to classes $\mathcal{P}_n(0, -R_{-1}(G))$ and $\mathcal{P}_{n-2}(0, 1 - R_{-1}(G_{r,s}))$ by Lemma 3.1 and Lemma 3.2, respectively, where a_1 and a_2 are fixed real numbers.

In [15], the following inequalities were presented for $P_{R(G)}(x)$, where eigenvalues are $\rho_1 \geq \rho_2 \geq \dots \geq \rho_n$:

$$\bar{x} + \frac{1}{n} \sqrt{\frac{\Delta}{n-1}} \leq \rho_1 \leq \bar{x} + \frac{1}{n} \sqrt{(n-1)\Delta}, \quad (16)$$

$$\bar{x} - \frac{1}{n} \sqrt{\frac{(i-1)\Delta}{n-i+1}} \leq \rho_i \leq \bar{x} + \frac{1}{n} \sqrt{\frac{(n-1)\Delta}{i}}, \quad \text{where } i = 2, 3, \dots, n-1, \quad (17)$$

$$\bar{x} - \frac{1}{n} \sqrt{(n-1)\Delta} \leq \rho_n \leq \bar{x} - \frac{1}{n} \sqrt{\frac{\Delta}{n-1}}, \quad (18)$$

where

$$\bar{x} = \frac{1}{n} \sum_{i=1}^n \rho_i = 0 \text{ and } \Delta = n \sum_{i=1}^n \rho_i^2 - \left(\sum_{i=1}^n \rho_i \right)^2 = 2nR_{-1}(G).$$

Now, let us adapt inequalities (16), (17) and (18) to arbitrary connected bipartite graphs.

Theorem 3.1 *Let $G_{r,s}$ be a connected bipartite graph with $n \geq 4$ vertices with the Randić polynomial $P_{R(G_{r,s})}(x)$. Then, the following inequality holds:*

$$\frac{3n-8}{2n-6} \leq R_{-1}(G_{r,s}) \leq \frac{n^2-5n+8}{2}.$$

Proof. Applying inequality (16) on the factor

$$(x^{n-2} + a_1x^{n-3} + a_2x^{n-4} + a_3x^{n-5} + \dots + a_{n-2})$$

of $P_{R(G_{r,s})}(x)$ reveals the following inequality:

$$\frac{1}{n-2} \sqrt{\frac{(n-2)(2R_{-1}(G_{r,s})-2)}{n-3}} \leq \rho_1 \leq \frac{1}{n-2} \sqrt{(n-3)(n-2)(2R_{-1}(G_{r,s})-2)}.$$

Then, a lower bound and an upper bound of $R_{-1}(G_{r,s})$ are obtained as follows:

$$(n-2)\rho_1 \leq \sqrt{(n-3)(n-2)(2R_{-1}(G_{r,s})-2)},$$

$$(n-2)^2\rho_1^2 \leq (n-3)(n-2)(2R_{-1}(G_{r,s})-2),$$

$$\frac{(n-2)\rho_1^2}{2(n-3)} + 1 \leq R_{-1}(G_{r,s}),$$

$$\frac{3n-8}{2n-6} \leq R_{-1}(G_{r,s}).$$

$$\sqrt{\frac{(n-2)(2R_{-1}(G_{r,s})-2)}{n-3}} \leq (n-2)\rho_1,$$

$$\frac{(n-2)(2R_{-1}(G_{r,s})-2)}{n-3} \leq (n-2)^2\rho_1^2,$$

$$R_{-1}(G_{r,s}) \leq \frac{(n-3)(n-2)\rho_1^2}{2} + 1,$$

$$R_{-1}(G_{r,s}) \leq \frac{n^2-5n+8}{2}. \square$$

Remark 3.1 *The lower bound*

$$R_{-1}(G_{r,s}) \geq \frac{3n-8}{2n-6} \quad (19)$$

is better than the lower bound

$$R_{-1}(G_{r,s}) \geq \frac{n}{2(n-1)} \quad (20)$$

from [8].

The same result in the Theorem 3.1 is obtained by using inequality (18). Furthermore, applying inequality (17) on the factor

$$(x^{n-2} + a_1x^{n-3} + a_2x^{n-4} + a_3x^{n-5} + \cdots + a_{n-2})$$

of $P_{R(G_{r,s})}(x)$ leads to the following corollary:

Corollary 3.1 *Let $G_{r,s}$ be a connected bipartite graph with $r+s=n \geq 4$ vertices with the Randić polynomial $P_{R(G_{r,s})}(x)$ and eigenvalues $\rho_1 = 1 \geq \rho_2 \geq \cdots \geq \rho_n = -1$.*

For each ρ_i satisfying the condition $0 \leq \rho_i$ except ρ_1 and ρ_n , the following inequality holds:

$$\frac{i(n-2)\rho_i^2}{2(n-i)} + 1 \leq R_{-1}(G_{r,s}). \quad (21)$$

For each ρ_i satisfying the condition $0 \leq \rho_i \leq \sqrt{\frac{(i-1)(2R_{-1}(G_{r,s})-2)}{(n-2)(n-i+1)}}$ except ρ_1 and ρ_n , the following inequality holds:

$$R_{-1}(G_{r,s}) \leq \frac{(n-2)(n-i+1)\rho_i^2}{2(i-1)} + 1. \quad (22)$$

For each ρ_i satisfying the condition $0 \leq \rho_i \leq \sqrt{\frac{(i-1)(2R_{-1}(G_{r,s})-2)}{(n-2)(n-i+1)}}$ except ρ_1 and ρ_n , the following inequality holds:

$$\frac{(n-2)(n-i+1)\rho_i^2}{2(i-1)} + 1 \leq R_{-1}(G_{r,s}). \quad (23)$$

For each ρ_i satisfying the condition $\rho_i \leq 0$ except ρ_1 and ρ_n , the following inequality holds:

$$\frac{(n-2)(n-i+1)\rho_i^2}{2(i-1)} + 1 \leq R_{-1}(G_{r,s}). \quad (24)$$

For each ρ_i satisfying the condition $-\sqrt{\frac{(n-i)(2R_{-1}(G_{r,s})-2)}{i(n-2)}} \leq \rho_i \leq 0$ except ρ_1 and ρ_n , the following inequality holds:

$$\frac{i(n-2)\rho_i^2}{2(n-i)} + 1 \leq R_{-1}(G_{r,s}). \quad (25)$$

For each ρ_i satisfying the condition $\rho_i \leq -\sqrt{\frac{(n-i)(2R_{-1}(G_{r,s})-2)}{i(n-2)}} \leq 0$ except ρ_1 and ρ_n , the following inequality holds:

$$R_{-1}(G_{r,s}) \leq \frac{i(n-2)\rho_i^2}{2(n-i)} + 1. \quad (26)$$

4 Remarks and conclusions

To summarize, we have seen that if $\sum_{i \sim j} \mathcal{F}(d_i, d_j)^2$ is a topological index, we can obtain some bounds of the $\sum_{i \sim j} \mathcal{F}(d_i, d_j)^2$ by means of the eigenvalues of the characteristic polynomial of the matrix $A_{\mathcal{F}}$ with entries $a_{ij} = a_{ji} = \mathcal{F}(d_i, d_j)$ for adjacent vertices i and j .

If $\sum_{i \sim j} \mathcal{F}(d_i, d_j)$ corresponds to a topological index already found in the literature instead of $\sum_{i \sim j} \mathcal{F}(d_i, d_j)^2$, then the eigenvalues of the polynomial that is the characteristic polynomial of the matrix $A_{\mathcal{F}}$ with entries $a_{ij} = a_{ji} = \sqrt{\mathcal{F}(d_i, d_j)}$ for adjacent vertices i and j can be used to get some bounds of $\sum_{i \sim j} \sqrt{\mathcal{F}(d_i, d_j)} = \sum_{i \sim j} \mathcal{F}(d_i, d_j)$.

For example, some topological indices can be considered for further research:

- For $\mathcal{F}(d_i, d_j) = \sqrt{d_i^2 + d_j^2}$, $TI = SO$ is the Sombor index [5] and $a_2 = -F$, by Lemma 2.1 for the corresponding characteristic polynomial of matrix $A_{\mathcal{F}}$, where F is the Forgotten topological index defined as $F = \sum_{i=1}^n d_i^3 = \sum_{i \sim j} (d_i^2 + d_j^2)$ [4].
- For $\mathcal{F}(d_i, d_j) = \sqrt{d_i d_j}$, $TI = RR$ is the reciprocal Randić index [6] and $a_2 = -M_2$, by Lemma 2.1 for the corresponding characteristic polynomial of matrix $A_{\mathcal{F}}$, where M_2 is the second Zagreb index defined as $M_2 = \sum_{i \sim j} d_i d_j$ [7].
- For $\mathcal{F}(d_i, d_j) = (d_i d_j)^\alpha$, $\alpha \in \mathbb{R}$, $TI = R_\alpha$ is the general Randić index (here we also use the term "general" but this index is also known as the general product-connectivity index [2]) and $a_2 = -R_{2\alpha}$.

- For $\mathcal{F}(d_i, d_j) = \frac{1}{\sqrt{d_i + d_j}}$, $TI = SCI$ is the sum-connectivity index [13] and $a_2 = -\chi_{-1}$ by Lemma 2.1 for the corresponding characteristic polynomial of matrix $A_{\mathcal{F}}$, where χ_{-1} the general sum-connectivity index [14].

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