

Solvability of a Mixed Problem for an Integro-Differential Equation with the Hilfer Type Fractional Analogue of the Barenblatt–Zhel'tov–Kochina Operator

Tursun K. Yuldashev, Bakhtiyor J. Kadirkulov, Aygul A. Matchanova

Abstract: In the article a partial integro-differential equation with degenerate kernel and Hilfer fractional operator is considered. The issues of unique classical solvability and construction of a solution to a mixed problem for a homogeneous integro-differential equation containing a Hilfer fractional analogue of the Barenblatt–Zhel'tov–Kochina operator are studied. The Fourier series method based on the separation of variables was used. By the aid of the Mittag–Leffler function is obtained a countable system. Sufficient coefficient conditions for unique classical solvability of the mixed problem are established. The absolute and uniform convergence of the obtained series is proved.

Keywords: integro-differential equation, Hilfer fractional analog of the Barenblatt–Zhel'tov–Kochina operator, degenerate kernel, regular solvability.

1 Introduction

The main provisions and equations of the theory of non-stationary filtration in fractured-pore formations are formulated in the work of G. I. Barenblatt, Yu. P. Zhel'tov and I. N. Kochin [19] (see also [22]) and then developed by many authors [34, 26, 27, 48, 51, 52]. The theory and applications of fractional calculus have been rapidly developed by many authors (see, for example, [31, 57, 35, 36, 42, 37, 47, 50, 53]). Fractional order ordinary differential equations are studied in many works (see, for example [15, 16, 29, 33, 45, 61, 62]). To study periodic solutions of fractional order ordinary differential equations are devoted many works of large number of mathematicians (see, for example, [3, 13, 14, 38, 63, 64, 65]). The generalized integro-differentiation operator is a continuous interpolation of the well-known fractional order differentiation operators of Riemann–Liouville and Caputo, which describe diffusion processes [57, vol. 1, pp. 47–85]. A physical and engineering interpretation of the generalized fractional

Manuscript received April 17, 2024; accepted November 5, 2024

Tursun K. Yuldashev (ORCID 0000-0002-9346-5362) is Professor of Tashkent State University of Economics, Tashkent, Uzbekistan;

Bakhtiyor J. Kadirkulov (ORCID 0000-0001-8937-0523) is Associate Professor of Tashkent State University of Oriental Studies, Tashkent, Uzbekistan;

Aygul A. Matchanova (ORCID 0009-0007-6369-9129) is PhD student of V. I. Romanovskii Institute of Mathematics, Tashkent, Uzbekistan

<https://doi.org/10.46793/SPSUNP2402.125Y>

operator is given in [57, vol. pp. 4–8]. Note that boundary value conditions of the type (3) (see below) take place in modeling problems of the flow around a profile by a subsonic velocity stream with a supersonic zone. Different mixed and boundary value problems for partial differential and integro-differential equations were studied in the works of many authors, in particular, in [1, 2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 23, 24, 25, 28, 30, 41, 43, 44, 46, 59, 60, 66, 67, 68, 69, 70, 71, 72]. Sturm–Liouville and associated problems are considered by many authors (see, for examples, [18, 20, 21, 32, 39, 40, 49, 54, 55, 56, 58]).

In our work, the problem of small denominators does not arise. We consider a mixed problem for a linear integro-differential equation with Hilfer operator of fractional integro-differentiation. The Fourier method of separation of variables is used taking into account the features of the fractional integro-differentiation operator and linearity of the problem. The obtained spectral problem is solved. Thus, in the domain $\Omega \equiv (0, T) \times (0, 1)$ we consider the equation

$$\left(D^{\alpha, \gamma} - D^{\alpha, \gamma} \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial x^2} \right) U(t, x) = v \int_0^T K(t, s) U(s, x) ds \quad (1)$$

with mixed conditions

$$\lim_{t \rightarrow +0} J_{0t}^{1-\gamma} U(t, x) = \varphi(x), \quad 0 \leq x \leq 1, \quad (2)$$

$$U(t, 0) = 0, \quad 0 \leq t \leq T, \quad (3)$$

$$U_x(t, 1) = U_x(t, x_0), \quad 0 \leq t \leq T, \quad 0 < x_0 < 1, \quad (4)$$

where $\varphi(x) \in C^6[0, 1]$,

$$\frac{d^n \varphi}{dx^n} \Big|_{x=0} = 0, \quad \frac{d^m \varphi(x)}{dx^m} \Big|_{x=1} = \frac{d^m \varphi(x)}{dx^m} \Big|_{x=x_0}, \quad n = 0, 2, 4, 6, \quad m = 1, 3, 5,$$

$K(t, s) = \sum_{r=1}^k t^r s^r$, $D^{\alpha, \gamma}$, $0 < \alpha \leq \gamma \leq 1$ is Hilfer integro-differential operator, which is defined by the formula $D^{\alpha, \gamma} = D_{0t}^{\alpha, \beta} = J_{0t}^{\beta(1-\alpha)} \frac{d}{dt} J_{0t}^{(1-\beta)(1-\alpha)}$, v is real parameter, $0 \leq \beta \leq 1$, $\gamma = \alpha + \beta - \alpha\beta$, $0 < T < \infty$.

Problem. It is required to find unknown function $U(t, x)$, that satisfies the integro-differential equation (1), the initial value condition (2), the boundary value conditions (3), (4) and belongs to the class of functions

$$t^{1-\gamma} U \in C(\overline{\Omega}), \quad D^{\alpha, \gamma} U \in C(\Omega), \quad D^{\alpha, \gamma} U_{xx} \in C(\Omega), \quad U_{xx} \in C(\Omega), \quad (5)$$

where $\overline{\Omega} \equiv [0, T] \times [0, 1]$.

2 Eigenvalues and eigenfunctions

First, we consider homogeneous differential equation

$$D^{\alpha, \gamma} U(t, x) - D^{\alpha, \gamma} U_{xx}(t, x) - U_{xx}(t, x) = 0. \quad (6)$$

We will look for a non-trivial particular solution of this problem in the form

$$U(t, x) = u(t) \cdot \vartheta(x).$$

Substituting this solution into equation (6), we obtain

$$\frac{D^{\alpha,\gamma}u(t)}{D^{\alpha,\gamma}u(t)+u(t)} = \frac{\vartheta''(x)}{\vartheta(x)}.$$

Hence, equating second fraction into $(-\lambda)$, $\lambda \geq 0$, we obtain

$$\vartheta''(x) + \lambda \vartheta(x) = 0, \quad \lambda \geq 0. \quad (7)$$

From the conditions (3) and (4), we obtain conditions for the eigenvalues λ and eigenfunctions $\vartheta(x)$:

$$\vartheta(0) = 0, \quad \vartheta'(1) = \vartheta'(x_0), \quad 0 < x_0 < 1. \quad (8)$$

Solving the spectral problem (7), (8), we derive eigenvalues (see [17])

$$\lambda_0 = 0, \quad \lambda_{1,n} = \left(\frac{2n\pi}{1+x_0} \right)^2, \quad \lambda_{2,n} = \left(\frac{2n\pi}{1-x_0} \right)^2, \quad n \in \mathbb{N}. \quad (9)$$

Eigenfunctions, corresponding to the eigenvalues (9), have the forms

$$\vartheta_0(x) = x, \quad \vartheta_{i,n}(x) = \sin \sqrt{\lambda_{i,n}}x, \quad i = 1, 2, \quad n \in \mathbb{N}. \quad (10)$$

Condition A. Let x_0 be a rational number from the interval $(0, 1)$ such that $x_0 = \frac{p}{q}$, $p < q$, $q - p = 1$, p and q be coprime positive integers. If condition A is satisfying, then solving problem (7), (8), instead (9) we obtain

$$\lambda_0 = 0, \quad \lambda_{1,n} = \left(\frac{2qn\pi}{p+q} \right)^2, \quad \lambda_{2,m} = (2qm\pi)^2, \quad n, m \in \mathbb{N}, \quad n \neq m(p+q). \quad (11)$$

These eigenvalues correspond to eigenfunctions in (10). For each value of $\lambda_{2,m}$, there also exist associated functions of the form

$$\tilde{\vartheta}_{2,m}(x) = x \cos \sqrt{\lambda_{2,m}}x. \quad (12)$$

Along with problem (7), (8), we also consider the problem adjoint to it. It is not difficult to determine that the following problem will be adjoint to problem (7), (8):

$$\omega''(x) + \lambda \omega(x) = 0, \quad \lambda \geq 0, \quad x \in (0, x_0) \cup (x_0, 1), \quad (13)$$

$$\omega(0) = 0, \quad \omega'(1) = 0, \quad (14)$$

$$\omega'(x_0+0) = \omega'(x_0-0), \quad \omega(x_0+0) - \omega(x_0-0) = \omega(1). \quad (15)$$

Problem (13)–(15) also has eigenvalues of the form (11). Solving this problem, it is not difficult to see that the eigenfunctions have the form

$$\{\omega_0(x); \omega_{1,n}(x); \omega_{2,m}(x)\}, \quad n, m \in \mathbb{N}, \quad n \neq m(p+q). \quad (16)$$

where

$$\omega_0(x) = \begin{cases} 0, & x \in [0, x_0), \\ \frac{2}{1-x_0^2}, & x \in (x_0, 1], \end{cases} \quad \omega_{1,n}(x) = \begin{cases} \frac{4 \sin \sqrt{\lambda_{1,n}}x}{1+x_0}, & x \in [0, x_0), \\ \frac{2 \cos \sqrt{\lambda_{1,n}}(1-x)}{(1+x_0) \sin \sqrt{\lambda_{1,n}}}, & x \in (x_0, 1], \end{cases}$$

$$\omega_{2,m}(x) = \begin{cases} 0, & x \in [0, x_0), \\ \frac{4 \cos \sqrt{\lambda_{2,m} x}}{1-x_0}, & x \in (x_0, 1]. \end{cases}$$

Reasoning similarly for the problem (11)–(14), we obtain the same results. We note that there also exist associated functions of the form

$$\tilde{\omega}_{2,m}(x) = \begin{cases} \frac{4 \sin \sqrt{\lambda_{2,m} x}}{1-x_0}, & x \in [0, x_0), \\ \frac{4(1-x) \sin \sqrt{\lambda_{2,m} x}}{1-x_0^2}, & x \in (x_0, 1]. \end{cases} \quad (17)$$

We note that systems of functions (10), (12) and (16), (17) are biorthonormal in $L_2[0, 1]$:

$$(\vartheta_0(x), \omega_0(x)) = 1, \quad (\vartheta_0(x), \omega_{1,n}(x)) = (\vartheta_0(x), \omega_{2,m}(x)) = (\vartheta_0(x), \tilde{\omega}_{2,m}(x)) = 0,$$

$$(\vartheta_{1,n}(x), \omega_{1,k}(x)) = \begin{cases} 1, & n = k, \\ 0, & n \neq k, \end{cases}$$

$$(\vartheta_{1,n}(x), \omega_0(x)) = (\vartheta_{1,n}(x), \omega_{2,m}(x)) = (\vartheta_{1,n}(x), \tilde{\omega}_{2,m}(x)) = 0,$$

$$(\vartheta_{2,m}(x), \tilde{\omega}_{2,k}(x)) = \begin{cases} 1, & m = k, \\ 0, & m \neq k, \end{cases}$$

$$(\vartheta_{2,m}(x), \omega_0(x)) = (\vartheta_{2,m}(x), \omega_{1,n}(x)) = (\vartheta_{2,m}(x), \omega_{2,k}(x)) = 0,$$

$$(\tilde{\vartheta}_{2,m}(x), \omega_{2,k}(x)) = \begin{cases} 1, & m = k, \\ 0, & m \neq k, \end{cases}$$

$$(\tilde{\vartheta}_{2,m}(x), \omega_0(x)) = (\tilde{\vartheta}_{2,m}(x), \tilde{\omega}_{2,k}(x)) = 0,$$

where $(\cdot, \cdot)$ is the inner product in $L_2[0, 1]$.

It is not difficult to see that if the condition A is satisfying, then the systems of root functions of problems (7), (8) and (13)–(15) form a Riesz basis in $L_2[0, 1]$. Taking into account the formulas (10) and (16), we look for a solution

$$U(t, x) = U_0(t, x) + U_1(t, x) + U_2(t, x) + \tilde{U}_2(t, x) \quad (18)$$

to the problem (1)–(4) in the following Fourier series:

$$U(t, x) = u_0(t) \vartheta_0(x) + \sum_{n=1}^{\infty} u_{1,n}(t) \vartheta_{1,n}(x) + \sum_{m=1}^{\infty} [u_{2,m}(t) \vartheta_{2,m}(x) + \tilde{u}_{2,m}(t) \tilde{\vartheta}_{2,m}(x)], \quad (19)$$

where

$$u_0(t) = \int_0^1 U_0(t, y) \omega_0(y) dy, \quad u_{1,n}(t) = \int_0^1 U_1(t, y) \omega_{1,n}(y) dy,$$

$$u_{2,m}(t) = \int_0^1 U_2(t, y) \tilde{\omega}_{2,m}(y) dy, \quad \tilde{u}_{2,m}(t) = \int_0^1 \tilde{U}_2(t, y) \omega_{2,m}(y) dy.$$

Here “*” means that the sum is taken over $n \in \mathbb{N}$, different from $k(q + p)$, $k \in \mathbb{N}$.

3 Reducing the solution to problem (1)–(4) to a countable system of non-linear integral equations

Let the condition A be satisfied and a function $U(t, x)$ be a solution to the mixed problem (1)–(4). Then, applying representation (19) into equation (1) and taking (10) and (18) into account, we obtain

$$\begin{aligned} & D^{\alpha, \gamma} \left\{ x u_0(t) + \sum_{n=1}^{\infty*} u_{1,n}(t) \vartheta_{1,n}(x) + \sum_{m=1}^{\infty} [u_{2,m}(t) \vartheta_{2,m}(x) + \tilde{u}_{2,m}(t) \tilde{\vartheta}_{2,m}(x)] + \right. \\ & + \sum_{n=1}^{\infty*} \lambda_{1,n} u_{1,n}(t) \vartheta_{1,n}(x) + \sum_{m=1}^{\infty} \left[\lambda_{2,m} u_{2,m}(t) \vartheta_{2,m}(x) + \tilde{u}_{2,m}(t) \left(2\sqrt{\lambda_{2,m}} \vartheta_{2,m}(x) + \lambda_{2,m} \tilde{\vartheta}_{2,m}(x) \right) \right] \Big\} + \\ & + \sum_{n=1}^{\infty*} \lambda_{1,n} u_{1,n}(t) \vartheta_{1,n}(x) + \sum_{m=1}^{\infty} \left[\lambda_{2,m} u_{2,m}(t) \vartheta_{2,m}(x) + \tilde{u}_{2,m}(t) \left(2\sqrt{\lambda_{2,m}} \vartheta_{2,m}(x) + \lambda_{2,m} \tilde{\vartheta}_{2,m}(x) \right) \right] = \\ & = v \int_0^T K(t, s) \left[x u_0(s) + \sum_{n=1}^{\infty*} u_{1,n}(s) \vartheta_{1,n}(x) + \sum_{m=1}^{\infty} [u_{2,m}(s) \vartheta_{2,m}(x) + \tilde{u}_{2,m}(s) \tilde{\vartheta}_{2,m}(x)] \right] ds. \end{aligned}$$

Hence, taking (16) and (17) into account, we obtain

$$D^{\alpha, \gamma} u_0(t) = v \int_0^T K(t, s) u_0(s) ds, \quad (20)$$

$$D^{\alpha, \gamma} u_{1,n}(t) + \mu_{1,n} u_{1,n}(t) = \frac{v}{1 + \lambda_{1,n}} \int_0^T K(t, s) u_{1,n}(s) ds, \quad (21)$$

$$\begin{aligned} & D^{\alpha, \gamma} u_{2,m}(t) + \mu_{2,m} u_{2,m}(t) = \\ & = -\frac{2\sqrt{\lambda_{2,m}}}{1 + \lambda_{2,m}} (D^{\alpha, \gamma} \tilde{u}_{2,m}(t) + \tilde{u}_{2,m}(t)) + \frac{v}{1 + \lambda_{2,m}} \int_0^T K(t, s) u_{2,m}(s) ds, \end{aligned} \quad (22)$$

$$D^{\alpha, \gamma} \tilde{u}_{2,m}(t) + \mu_{2,m} \tilde{u}_{2,m}(t) = \frac{v}{1 + \lambda_{2,m}} \int_0^T K(t, s) \tilde{u}_{2,m}(s) ds, \quad (23)$$

$$\mu_{1,n} = \frac{\lambda_{1,n}}{1 + \lambda_{1,n}}, \quad \mu_{2,m} = \frac{\lambda_{2,m}}{1 + \lambda_{2,m}},$$

$\lambda_{1,n}$ and $\lambda_{2,m}$ ($n, m \in \mathbb{N}$, $n \neq m(p+q)$) defined from (11).

The solving method for all fractional integro-differential equations (20), (21) and (23) is the same. So, write the function $\varphi(x)$ as

$$\varphi(x) = \varphi_0(x) + \varphi_1(x) + \varphi_2(x) + \tilde{\varphi}_2(x).$$

Then we show the scheme of solving only for the equations (20) and (21). To solve the countable system of integro-differential equations (20) and (21), from (2) we determine the initial value condition

$$\lim_{t \rightarrow +0} J_{0t}^{1-\gamma} u_0(t) = \lim_{t \rightarrow +0} J_{0t}^{1-\gamma} \int_0^1 U_0(t, y) \omega_0(y) dy = \int_0^1 \varphi_0(x) \omega_0(y) dy = \varphi_0, \quad (24)$$

$$\lim_{t \rightarrow +0} J_{0t}^{1-\gamma} u_{1,n}(t) = \lim_{t \rightarrow +0} J_{0t}^{1-\gamma} \int_0^1 U_1(t, y) \omega_{1,n}(y) dy = \int_0^1 \varphi_1(x) \omega_{1,n}(y) dy = \varphi_{1,n}, \quad (25)$$

$$\lim_{t \rightarrow +0} J_{0t}^{1-\gamma} u_{2,m}(t) = \lim_{t \rightarrow +0} J_{0t}^{1-\gamma} \int_0^1 U_2(t,y) \tilde{\omega}_{2,m}(y) dy = \int_0^1 \varphi_2(x) \tilde{\omega}_{2,m}(y) dy = \varphi_{2,m}, \quad (26)$$

$$\lim_{t \rightarrow +0} J_{0t}^{1-\gamma} \tilde{u}_{2,m}(t) = \lim_{t \rightarrow +0} J_{0t}^{1-\gamma} \int_0^1 \tilde{U}_2(t,y) \omega_{2,m}(y) dy = \int_0^1 \varphi_2(x) \omega_{2,m}(y) dy = \tilde{\varphi}_{2,m}. \quad (27)$$

First, we solve the equation (20). In this order we rewrite the equation (20) in the form

$$D^{\alpha,\gamma} u_0(t) = g_0(t), \quad (28)$$

where

$$g_0(t) = v \sum_{r=1}^k t^r \tau_{0,r}, \quad (29)$$

$$\tau_{0,r} = \int_0^T s^r u_{0,r}(s) ds. \quad (30)$$

Applying the operator J_{0t}^α to both sides of the equation (28), we have

$$u_0(t) = \frac{C_0}{\Gamma(\gamma)} t^{\gamma-1} + \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\alpha-1} g_0(s) ds,$$

where C_0 is arbitrary constant.

Using initial value condition (24) and designation (29), we represent the solution of the system (28) in the form

$$u_0(t) = \frac{\varphi_0}{\Gamma(\gamma)} t^{\gamma-1} + \frac{v}{\Gamma(\gamma)} \sum_{r=1}^k \tau_{0,r} \int_0^t (t-s)^{\alpha-1} s^r ds. \quad (31)$$

Substituting the representation (31) into (30), we obtain

$$\tau_{0,r} = \frac{\varphi_0}{\Gamma(\gamma)} \int_0^T s^{\gamma+r-1} ds + v \sum_{j=1}^k \tau_{0,j} \frac{1}{\Gamma(\gamma)} \int_0^T s^r \int_0^s \theta^r (s-\theta)^{\alpha-1} d\theta ds$$

or we obtain a linear system of algebraic equations (LSAE)

$$\tau_{0,r} = v \sum_{j=1}^k \tau_{0,j} A_{0,r,j} + B_{0,r}, \quad r = \overline{1, k}, \quad (32)$$

where

$$A_{0,r,j} = \frac{1}{\Gamma(\gamma)} \int_0^T s^r \int_0^s \theta^j (s-\theta)^{\alpha-1} d\theta ds, \quad B_{0,r} = \frac{\varphi_0}{\Gamma(\gamma)} \frac{T^{r+\gamma}}{r+\gamma}.$$

We consider the following matrices for our further calculations:

$$Z_0(v) = \begin{pmatrix} 1 - vA_{0,1,1} & vA_{0,1,2} & \dots & vA_{0,1,k} \\ vA_{0,2,1} & 1 - vA_{0,2,2} & \dots & vA_{0,2,k} \\ \dots & \dots & \dots & \dots \\ vA_{0,k,1} & vA_{0,k,2} & \dots & 1 - vA_{0,k,k} \end{pmatrix},$$

$$Z_{0,r}(v) = \begin{pmatrix} 1 - vA_{0,1,1} & \dots & vA_{0,1,(r-1)} & B_{0,1} & vA_{0,1,(r+1)} & \dots & vA_{0,1,k} \\ vA_{0,2,1} & \dots & vA_{0,2,(r-1)} & B_{0,2} & vA_{0,2,(r+1)} & \dots & vA_{0,2,k} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ vA_{0,k,1} & \dots & vA_{0,k,(r-1)} & B_{0,k} & vA_{0,k,(r+1)} & \dots & 1 - vA_{0,k,k} \end{pmatrix}.$$

LSAE (32) is uniquely solvable for any finite right-hand sides, if the following non-degeneracy condition for the Fredholm determinant is satisfied

$$\Delta_0(v) = \det Z_0(v) \neq 0. \quad (33)$$

The determinant (33) $\Delta_0(v)$ is a polynomial with respect to v of degree not higher k . The equation $\Delta_0(v) = 0$ has at most k different real roots. We denote them by θ_p ($p = \overline{1, k_p}$, $1 \leq k_p \leq k$). Then $v = v_p = \theta_p$ called as the irregular values of the spectral parameter v . Other values of the spectral parameter $v \neq \theta_p$ are called as the regular.

For regular values of the spectral parameter v the solution of the LSAE (32) has the form

$$\tau_{0,r} = \frac{\Delta_{0,r}(v)}{\Delta_0(v)}, \quad r = \overline{1, k}, \quad (34)$$

where $\Delta_{0,r}(v) = \det Z_{0,r}(v)$. Substituting the representation of solution (34) of the LSAE (32) into representation (31) of the $u_0(t)$, we derive

$$u_0(t) = \frac{\varphi_0}{\Gamma(\gamma)} t^{\gamma-1} + \frac{v}{\Gamma(\gamma)} \sum_{r=1}^k \frac{\Delta_{0,r}(v)}{\Delta_0(v)} \int_0^t (t-s)^{\alpha-1} s^r ds.$$

Hence, we obtain

$$t^{1-\gamma} u_0(t) = \frac{\varphi_0}{\Gamma(\gamma)} + \frac{v}{\Gamma(\gamma)} \sum_{r=1}^k \frac{\Delta_{0,r}(v)}{\Delta_0(v)} t^{1-\gamma} \int_0^t (t-s)^{\alpha-1} s^r ds. \quad (35)$$

From (35) we have

$$t^{1-\gamma} U_0(t, x) = \vartheta_0(x) \left[\frac{\varphi_0}{\Gamma(\gamma)} + \frac{v}{\Gamma(\gamma)} \sum_{r=1}^k \frac{\Delta_{0,r}(v)}{\Delta_0(v)} t^{1-\gamma} \int_0^t (t-s)^{\alpha-1} s^r ds \right]. \quad (36)$$

Now we solve the countable system (CS) of equations (21). The CS of integro-differential equations (21) we rewrite as

$$D^{\alpha, \gamma} u_{1,n}(t) = -\mu_{1,n} u_{1,n}(t) + g_{1,n}(t), \quad (37)$$

where

$$g_{1,n}(t) = \frac{v}{1 + \lambda_{1,n}} \sum_{r=1}^k t^r \tau_{1,n,r}, \quad (38)$$

$$\tau_{1,n,r} = \int_0^T s^r u_{1,n,r}(s) ds. \quad (39)$$

Applying the operator J_{0t}^α to both sides of the equation (37), and taking into account the formula

$$J_{0t}^\gamma D_{0t}^\gamma u_{1,n}(t) = u_{1,n}(t) - \frac{1}{\Gamma(\gamma)} J_{0t}^{1-\gamma} u_{1,n}(t)|_{t=0} t^{\gamma-1},$$

we have

$$u_{1,n}(t) = \frac{C_1}{\Gamma(\gamma)} t^{\gamma-1} + J_{0t}^\alpha g_{1,n}(t) - \mu_{1,n} J_{0t}^\alpha u_{1,n}(t),$$

where C_1 is arbitrary constant.

Using initial value condition (25), we represent the solution of the countable system (37) in the form

$$u_{1,n}(t) = \frac{\varphi_{1,n}}{\Gamma(\gamma)} t^{\gamma-1} + J_{0+}^{\alpha} g_{1,n}(t) - \mu_{1,n} \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\mu_{1,n}(t-s)^{\alpha}) \left[\frac{\varphi_{1,n}}{\Gamma(\gamma)} s^{\gamma-1} + J_{0+}^{\alpha} g_{1,n}(s) \right] ds.$$

We will write this representation in the following form

$$u_{1,n}(t) = \varphi_{1,n} \left[\frac{t^{\gamma-1}}{\Gamma(\gamma)} - \frac{\mu_{1,n}}{\Gamma(\gamma)} \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\mu_{1,n}(t-s)^{\alpha}) s^{\gamma-1} ds \right] + J_{0+}^{\alpha} g_{1,n}(t) - \mu_{1,n} \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\mu_{1,n}(t-s)^{\alpha}) J_{0+}^{\alpha} g_{1,n}(s) ds. \quad (40)$$

In representation (40) we take into account the following relations

$$\begin{aligned} \frac{t^{\gamma-1}}{\Gamma(\gamma)} - \frac{\mu_{1,n}}{\Gamma(\gamma)} \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\mu_{1,n}(t-s)^{\alpha}) s^{\gamma-1} ds &= t^{\gamma-1} E_{\alpha,\gamma}(-\mu_{1,n} t^{\alpha}), \\ \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\mu_{1,n}(t-s)^{\alpha}) J_{0+}^{\alpha} g_{1,n}(s) ds &= \\ = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\mu_{1,n}(t-s)^{\alpha}) ds \int_0^s (s-\theta)^{\alpha-1} g_{1,n}(\theta) d\theta &= \\ = \frac{1}{\Gamma(\alpha)} \int_0^t g_{1,n}(s) ds \int_s^t (t-s)^{\alpha-1} (s-\theta)^{\alpha-1} E_{\alpha,\alpha}(-\mu_{1,n}(t-\theta)^{\alpha}) d\theta, \\ \int_s^t (t-s)^{\alpha-1} (s-\theta)^{\alpha-1} E_{\alpha,\alpha}(-\mu_{1,n}(t-\theta)^{\alpha}) d\theta &= \\ = \Gamma(\alpha) (t-s)^{2\alpha-1} E_{\alpha,2\alpha}(-\mu_{1,n}(t-s)^{\alpha}). \end{aligned}$$

By virtue of these relations, from the presentation (40) we obtain

$$u_{1,n}(t) = \varphi_{1,n} t^{\gamma-1} E_{\alpha,\gamma}(-\mu_{1,n} t^{\alpha}) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\mu_{1,n}(t-s)^{\alpha}) g_{1,n}(s) ds, \quad (41)$$

where $E_{\alpha,\gamma}(z)$ is Mittag-Leffler function

$$E_{\alpha,\gamma}(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(\alpha m + \gamma)}, \quad z, \alpha, \gamma \in \mathbb{C}, \quad \operatorname{Re}(\alpha) > 0.$$

In obtaining the equation we took into account the following representations

$$\begin{aligned} E_{\alpha,\gamma}(z) &= \frac{1}{\Gamma(\gamma)} + z E_{\alpha,\gamma+\alpha}(z), \quad \alpha > 0, \quad \gamma > 0, \\ \frac{1}{\Gamma(\alpha)} \int_0^z (z-t)^{\alpha-1} E_{\alpha,\gamma}(kt^{\alpha}) t^{\gamma-1} dt &= z^{\gamma+\alpha-1} E_{\alpha,\gamma+\alpha}(kz^{\alpha}), \quad \alpha > 0, \quad \gamma > 0. \end{aligned}$$

Taking (38) into account, the equation (41) we rewrite as

$$u_{1,n}(t) = \varphi_{1,n} t^{\gamma-1} E_{\alpha,\gamma}(-\mu_{1,n} t^{\alpha}) +$$

$$+ \frac{\nu}{1 + \lambda_{1,n}} \sum_{r=1}^k \tau_{1,n,r} \int_0^t s^r (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\mu_{1,n}(t-s)^\alpha) ds. \quad (42)$$

Substituting the presentation (42) into (39), we obtain

$$\begin{aligned} \tau_{1,n,r} = & \int_0^T s^r \left[\varphi_{1,n} s^{\gamma-1} E_{\alpha,\gamma}(-\mu_{1,n} s^\alpha) + \right. \\ & \left. + \frac{\nu}{1 + \lambda_{1,n}} \sum_{j=1}^k \tau_{1,n,j} \int_0^s \theta^j (s-\theta)^{\alpha-1} E_{\alpha,\alpha}(-\mu_{1,n}(s-\theta)^\alpha) d\theta \right] ds \end{aligned}$$

or we obtain a linear system of countable system of algebraic equations (LSCSAE)

$$\tau_{1,n,r} = \nu \sum_{j=1}^k \tau_{1,n,j} A_{1,n,r,j} + B_{1,n,r}, \quad r = \overline{1, k}, \quad (43)$$

where

$$\begin{aligned} A_{1,n,r,j} &= \frac{1}{1 + \lambda_{1,n}} \int_0^T s^r \left[\int_0^s \theta^j (s-\theta)^{\alpha-1} E_{\alpha,\alpha}(-\mu_{1,n}(s-\theta)^\alpha) d\theta \right] ds = \\ &= \frac{T^{\alpha+r+j+1}}{1 + \lambda_{1,n}} \sum_{l=0}^r \frac{(-1)^l r!}{(r-l)!} E_{\alpha,\alpha+j+2+l}(-\mu_{1,n} T^\alpha), \\ B_{1,n,r} &= \varphi_{1,n} \int_0^T s^{r+\gamma-1} E_{\alpha,\gamma}(-\mu_{1,n} s^\alpha) ds = \\ &= \varphi_{1,n} T^{\gamma+r} \sum_{l=0}^r \frac{(-1)^l r!}{(r-l)!} E_{\alpha,\gamma+1+l}(-\mu_{1,n} T^\alpha). \end{aligned} \quad (44)$$

We consider the following matrices

$$\begin{aligned} Z_{1,n}(\nu) &= \begin{pmatrix} 1 - \nu A_{1,1,1} & \nu A_{1,1,2} & \dots & \nu A_{1,1,k} \\ \nu A_{1,2,1} & 1 - \nu A_{1,2,2} & \dots & \nu A_{1,2,k} \\ \dots & \dots & \dots & \dots \\ \nu A_{1,k,1} & \nu A_{1,k,2} & \dots & 1 - \nu A_{1,k,k} \end{pmatrix}, \\ Z_{1,n,r}(\nu) &= \begin{pmatrix} 1 - \nu A_{1,1,1} & \dots & \nu A_{1,1,(r-1)} & B_{1,1} & \nu A_{1,1,(r+1)} & \dots & \nu A_{1,1,k} \\ \nu A_{1,2,1} & \dots & \nu A_{1,2,(r-1)} & B_{1,2} & \nu A_{1,2,(r+1)} & \dots & \nu A_{1,2,k} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \nu A_{1,k,1} & \dots & \nu A_{1,k,(r-1)} & B_{1,k} & \nu A_{1,k,(r+1)} & \dots & 1 - \nu A_{1,k,k} \end{pmatrix}, \end{aligned}$$

where $A_{1,r,j} = A_{1,n,r,j}$, $B_{1,r} = B_{1,n,r}$, $r = \overline{1, k}$.

LSCSAE (43) is uniquely solvable for any finite right-hand sides, if the following non-degeneracy condition for the Fredholm determinant is satisfied

$$\Delta_{1,n}(\nu) = \det Z_{1,n}(\nu) \neq 0. \quad (45)$$

The determinant $\Delta_{1,n}(\nu)$ in (45), as we indicated above, is a polynomial with respect to ν of degree not higher k . The equation $\Delta_{1,n}(\nu) = 0$ has at most k different real roots. We denote

them by θ_p ($p = \overline{1, k_p}$, $1 \leq k_p \leq k$). Then $v = v_{n+p} = \theta_p$ are called as the irregular values of the spectral parameter v . Other values of the spectral parameter $v \neq \theta_p$, for which is true

$$\sum_{n=1}^{\infty} \frac{1}{|\Delta_{1,n}(v)|} < \infty,$$

are called – regular.

For regular values of the spectral parameter v the solution of the LSCSAE (43) has the form

$$\tau_{1,n,r} = \frac{\Delta_{1,n,r}(v)}{\Delta_{1,n}(v)}, \quad r = \overline{1, k}, \quad (46)$$

where $\Delta_{1,n,r}(v) = \det Z_{1,n,r}(v)$. Substituting the presentation of solution (46) of the LSCSAE (43) into representation (42) of the Fourier coefficients $u_{1,n}(t)$, we derive

$$\begin{aligned} u_{1,n}(t) &= \varphi_{1,n} t^{\gamma-1} E_{\alpha,\gamma}(-\mu_{1,n} t^{\alpha}) + \\ &+ \frac{v}{1 + \lambda_{1,n}} \sum_{r=1}^k \frac{\Delta_{1,n,r}(v)}{\Delta_{1,n}(v)} \int_0^t s^r (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\mu_{1,n}(t-s)^{\alpha}) ds. \end{aligned} \quad (47)$$

Taking

$$\begin{aligned} \int_0^t t^{1-\gamma} s^r (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\mu_{1,n}(t-s)^{\alpha}) ds &= t^{1-\gamma} \int_0^t s^r (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\mu_{1,n}(t-s)^{\alpha}) ds = \\ &= r! \cdot t^{1-\gamma+\alpha+r} E_{\alpha,\alpha+r+1}(-\mu_{1,n} t^{\alpha}) = \frac{r! \cdot t^{1-\gamma+r}}{\mu_{1,n}} \left(\frac{1}{r!} - E_{\alpha,r+1}(-\mu_{1,n} t^{\alpha}) \right) \end{aligned}$$

into account, instead of the countable system of equation (47) we consider the following CS

$$\begin{aligned} t^{1-\gamma} u_{1,n}(t) &= \varphi_{1,n} E_{\alpha,\gamma}(-\mu_{1,n} t^{\alpha}) + \\ &+ \frac{1}{1 + \lambda_{1,n}} \frac{v}{\mu_{1,n}} \sum_{r=1}^k \frac{\Delta_{1,n,r}(v)}{\Delta_{1,n}(v)} r! \cdot t^{1-\gamma+r} \left(\frac{1}{r!} - E_{\alpha,r+1}(-\mu_{1,n} t^{\alpha}) \right), \end{aligned} \quad (48)$$

where

$$\mu_{1,n} = \frac{\lambda_{1,n}}{1 + \lambda_{1,n}}, \quad \lambda_{1,n} = \left(\frac{2qn\pi}{p+q} \right)^2, \quad n = 1, 2, \dots,$$

$q - p = 1$, p and q be coprime positive integers.

From CS (48) we obtain the following Fourier series

$$\begin{aligned} t^{1-\gamma} U_1(t, x) &= \sum_{n=1}^{\infty*} \vartheta_{1,n}(x) \left[\varphi_{1,n} E_{\alpha,\gamma}(-\mu_{1,n} t^{\alpha}) + \right. \\ &\left. + \frac{v}{\lambda_{1,n}} \sum_{r=1}^k \frac{\Delta_{1,n,r}(v)}{\Delta_{1,n}(v)} t^{1-\gamma+r} \left(\frac{1}{r!} - E_{\alpha,r+1}(-\mu_{1,n} t^{\alpha}) \right) r! \right], \end{aligned} \quad (49)$$

where we took into account that

$$\frac{1}{1 + \lambda_{1,n}} \frac{v}{\mu_{1,n}} = \frac{1}{1 + \lambda_{1,n}} \frac{v}{\frac{\lambda_{1,n}}{1 + \lambda_{1,n}}} = \frac{v}{\lambda_{1,n}}.$$

By the same way, similarly to (48) and (49) for the CS of differential equation (23) with initial value condition (27) we obtain

$$t^{1-\gamma}\tilde{u}_{2,m}(t) = \tilde{\varphi}_{2,m}E_{\alpha,\gamma}(-\mu_{2,m}t^\alpha) + \frac{\nu}{\lambda_{2,m}} \sum_{r=1}^k \frac{\tilde{\Delta}_{2,m,r}(\nu)}{\tilde{\Delta}_{2,m}(\nu)} r! \cdot t^{1-\gamma+r} \left(\frac{1}{r!} - E_{\alpha,r+1}(-\mu_{2,m}t^\alpha) \right), \quad (50)$$

$$t^{1-\gamma}\tilde{U}_2(t,x) = \sum_{m=1}^{\infty} \tilde{\vartheta}_{2,m}(x) \left[\tilde{\varphi}_{2,m}E_{\alpha,\gamma}(-\mu_{2,m}t^\alpha) + \frac{\nu}{\lambda_{2,m}} \sum_{r=1}^k \frac{\tilde{\Delta}_{2,m,r}(\nu)}{\tilde{\Delta}_{2,m}(\nu)} t^{1-\gamma+r} \left(\frac{1}{r!} - E_{\alpha,r+1}(-\mu_{2,m}t^\alpha) \right) r! \right], \quad (51)$$

where

$$\sum_{n=1}^{\infty} \frac{1}{|\tilde{\Delta}_{2,m}(\nu)|} < \infty,$$

$$\tilde{\Delta}_{2,m}(\nu) = \begin{vmatrix} 1 - \nu\tilde{A}_{2,1,1} & \nu\tilde{A}_{2,1,2} & \dots & \nu\tilde{A}_{2,1,k} \\ \nu\tilde{A}_{2,2,1} & 1 - \nu\tilde{A}_{2,2,2} & \dots & \nu\tilde{A}_{2,2,k} \\ \dots & \dots & \dots & \dots \\ \nu\tilde{A}_{2,k,1} & \nu\tilde{A}_{2,k,2} & \dots & 1 - \nu\tilde{A}_{2,k,k} \end{vmatrix} \neq 0,$$

$$\tilde{\Delta}_{2,m,r}(\nu) = \begin{pmatrix} 1 - \nu\tilde{A}_{2,1,1} & \dots & \nu\tilde{A}_{2,1,(r-1)} & \tilde{B}_{2,1} & \nu\tilde{A}_{2,1,(r+1)} & \dots & \nu\tilde{A}_{2,1,k} \\ \nu\tilde{A}_{2,2,1} & \dots & \nu\tilde{A}_{2,2,(r-1)} & \tilde{B}_{2,2} & \nu\tilde{A}_{2,2,(r+1)} & \dots & \nu\tilde{A}_{2,2,k} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \nu\tilde{A}_{2,k,1} & \dots & \nu\tilde{A}_{2,k,(r-1)} & \tilde{B}_{2,k} & \nu\tilde{A}_{2,k,(r+1)} & \dots & 1 - \nu\tilde{A}_{2,k,k} \end{pmatrix},$$

$$\tilde{A}_{2,m,r,j} = \frac{T^{\alpha+r+j+1}}{1 + \lambda_{2,m}} \sum_{l=0}^r \frac{(-1)^l r!}{(r-l)!} E_{\alpha,\alpha+j+2+l}(-\mu_{2,m}T^\alpha),$$

$$\tilde{B}_{2,m,r} = \tilde{\varphi}_{2,m} T^{\gamma+r} \sum_{l=0}^r \frac{(-1)^l r!}{(r-l)!} E_{\alpha,\gamma+1+l}(-\mu_{2,m}T^\alpha),$$

$$\mu_{2,m} = \frac{\lambda_{2,m}}{1 + \lambda_{2,m}}, \quad \lambda_{2,m} = (2qm\pi)^2, \quad m = 1, 2, \dots$$

Now we will solve the CS of integro-differential equations (22) with initial value condition (26). We rewrite (22) as

$$D^{\alpha,\gamma}u_{2,m}(t) + \mu_{2,m}u_{2,m}(t) = \frac{\nu}{1 + \lambda_{2,m}} \int_0^T K(t,s)u_{2,m}(s)ds + h_m(t), \quad (52)$$

where

$$h_m(t) = -\frac{2\sqrt{\lambda_{2,m}}}{1 + \lambda_{2,m}} \left(D^{\alpha,\gamma}\tilde{u}_{2,m}(t) + \tilde{u}_{2,m}(t) \right)$$

or, taking (50) into account, we have

$$h_m(t) = -\frac{2\sqrt{\lambda_{2,m}}}{1 + \lambda_{2,m}} D^{\alpha,\gamma} \left[\tilde{\varphi}_{2,m}E_{\alpha,\gamma}(-\mu_{2,m}t^\alpha)t^{\gamma-1} + \right.$$

$$\begin{aligned}
& + \frac{\nu}{\lambda_{2,m}} \sum_{r=1}^k \frac{\tilde{\Delta}_{2,m,r}(\nu)}{\tilde{\Delta}_{2,m}(\nu)} t^r r! \left(\frac{1}{r!} - E_{\alpha, r+1}(-\mu_{2,m} t^\alpha) \right) \Big] - \\
& - \frac{2\sqrt{\lambda_{2,m}}}{1 + \lambda_{2,m}} \left[\tilde{\varphi}_{2,m} E_{\alpha, \gamma}(-\mu_{2,m} t^\alpha) t^{\gamma-1} + \frac{\nu}{\lambda_{2,m}} \sum_{r=1}^k \frac{\tilde{\Delta}_{2,m,r}(\nu)}{\tilde{\Delta}_{2,m}(\nu)} t^r r! \left(\frac{1}{r!} - E_{\alpha, r+1}(-\mu_{2,m} t^\alpha) \right) \right].
\end{aligned}$$

So, the functions $h_m(t)$ is known and for them is true the following estimate

$$|h_m(t)| \leq C_2 \sqrt{\sum_{m=1}^{\infty} \frac{1}{\lambda_{2,m}}} \|\tilde{\varphi}_2(x)\|_{L_2[0,1]}, \quad \lambda_{2,m} = (2qm\pi)^2, \quad 0 < C_2 = \text{const}. \quad (53)$$

The equation (52) we rewrite as

$$D^{\alpha, \gamma} u_{2,m}(t) = -\mu_{2,m} u_{2,m}(t) + g_{2,m}(t), \quad (54)$$

where

$$\begin{aligned}
g_{2,m}(t) &= \frac{\nu}{1 + \lambda_{2,m}} \sum_{r=1}^k t^r \tau_{2,m,r} + h_m(t), \\
\tau_{2,m,r} &= \int_0^T s^r u_{2,m,r}(s) ds.
\end{aligned} \quad (55)$$

By the similar way, as in the case of equation (21), for (54) we derive

$$\begin{aligned}
u_{2,m}(t) &= \varphi_{2,m} t^{\gamma-1} E_{\alpha, \gamma}(-\mu_{2,m} t^\alpha) + \int_0^t s^r (t-s)^{\alpha-1} E_{\alpha, \alpha}(-\mu_{2,m}(t-s)^\alpha) h_m(s) ds + \\
& + \frac{\nu}{1 + \lambda_{2,m}} \sum_{r=1}^k \tau_{2,m,r} \int_0^t s^r (t-s)^{\alpha-1} E_{\alpha, \alpha}(-\mu_{2,m}(t-s)^\alpha) ds.
\end{aligned} \quad (56)$$

Substituting the representation (56) into (55), we obtain a linear system of CS of algebraic equations (LSCSAE)

$$\tau_{2,m,r} = \nu \sum_{j=1}^k \tau_{2,m,j} A_{2,m,r,j} + B_{2,m,r}, \quad r = \overline{1, k}, \quad (57)$$

where

$$\begin{aligned}
A_{2,m,r,j} &= \frac{T^{\alpha+r+j+1}}{1 + \lambda_{2,m}} \sum_{l=0}^r \frac{(-1)^l r!}{(r-l)!} E_{\alpha, \alpha+j+2+l}(-\mu_{2,m} T^\alpha), \\
B_{2,m,r} &= \varphi_{2,m} T^{\gamma+r} \sum_{l=0}^r \frac{(-1)^l r!}{(r-l)!} E_{\alpha, \gamma+1+l}(-\mu_{2,m} T^\alpha) + \\
& + \int_0^t s^r (t-s)^{\alpha-1} E_{\alpha, \alpha}(-\mu_{2,m}(t-s)^\alpha) h_m(s) ds.
\end{aligned} \quad (58)$$

The determinant

$$\Delta_{2,m}(\nu) = \begin{vmatrix} 1 - \nu A_{2,1,1} & \nu A_{2,1,2} & \dots & \nu A_{2,1,k} \\ \nu A_{2,2,1} & 1 - \nu A_{2,2,2} & \dots & \nu A_{2,2,k} \\ \dots & \dots & \dots & \dots \\ \nu A_{2,k,1} & \nu A_{2,k,2} & \dots & 1 - \nu A_{2,k,k} \end{vmatrix}$$

is a polynomial with respect to v of degree not higher k . The equation $\Delta_{2,m}(v) = 0$ has at most k different real roots. We denote them by θ_p ($p = \overline{1, k_p}$, $1 \leq k_p \leq k$). Then $v = v_{m+p} = \theta_p$ called irregular values of the spectral parameter v . Other values of the spectral parameter $v \neq \theta_p$, for which is true

$$\sum_{m=1}^{\infty} \frac{1}{|\Delta_{2,m}(v)|} < \infty,$$

we call as the regular values.

For regular values of the spectral parameter v the solution of the LSCSAE (57) has the form

$$\tau_{2,m,r} = \frac{\Delta_{2,m,r}(v)}{\Delta_{2,m}(v)}, \quad r = \overline{1, k}, \quad (59)$$

where

$$\Delta_{2,m,r}(v) = \begin{vmatrix} 1 - vA_{2,1,1} & \dots & vA_{2,1,(r-1)} & B_{2,1} & vA_{2,1,(r+1)} & \dots & vA_{2,1,k} \\ vA_{2,2,1} & \dots & vA_{2,2,(r-1)} & B_{2,2} & vA_{2,2,(r+1)} & \dots & vA_{2,2,k} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ vA_{2,k,1} & \dots & vA_{2,k,(r-1)} & B_{2,k} & vA_{2,k,(r+1)} & \dots & 1 - vA_{2,k,k} \end{vmatrix}.$$

Substituting the presentation of solution (59) of the LSCSAE (57) into representation (56) of the Fourier coefficients $u_{1,n}(t)$, we derive

$$t^{1-\gamma}u_{2,m}(t) = \varphi_{2,m}E_{\alpha,\gamma}(-\mu_{2,m}t^\alpha) + \int_0^t t^{1-\gamma}s^r(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\mu_{2,m}(t-s)^\alpha)h_m(s)ds + \\ + \frac{v}{1+\lambda_{2,m}} \sum_{r=1}^k \frac{\Delta_{2,m,r}(v)}{\Delta_{2,m}(v)} \int_0^t t^{1-\gamma}s^r(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\mu_{2,m}(t-s)^\alpha)ds, \quad (60)$$

$$t^{1-\gamma}U_2(t, x) = \sum_{m=1}^{\infty} \vartheta_{2,m}(x) \left[\varphi_{2,m}E_{\alpha,\gamma}(-\mu_{2,m}t^\alpha) + \right. \\ \left. + \int_0^t t^{1-\gamma}s^r(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\mu_{2,m}(t-s)^\alpha)h_m(s)ds + \right. \\ \left. + \frac{v}{\lambda_{2,m}} \sum_{r=1}^k \frac{\Delta_{2,m,r}(v)}{\Delta_{2,m}(v)} t^{1-\gamma+r} \left(\frac{1}{r!} - E_{\alpha,r+1}(-\mu_{2,m}t^\alpha) \right) r! \right]. \quad (61)$$

4 Unique solvability of the mixed problem

We prove the uniqueness of the solution to the problem. To establish the uniqueness of the solution to the mixed problem (1)–(5), we show that with a homogeneous initial condition $[t^{1-\gamma}U(t, x)]_{t=0} \equiv 0$, this problem has only a trivial solution. Indeed, in condition (2) we assume that $\varphi(x) \equiv 0$. Then $\varphi_0 = \varphi_{1,n} = \varphi_{2,m} = \tilde{\varphi}_{2m} = 0$. From the representations (36), (49), (51) and (61) implies that

$$t^{1-\gamma}u_0(t) = \int_0^1 t^{1-\gamma}U_0(t, y)\omega_0(y)dy \equiv 0,$$

$$t^{1-\gamma}u_{1,n}(t) = \int_0^1 t^{1-\gamma}U_1(t, y)\omega_{1,n}(y)dy \equiv 0, \quad n = 1, 2, \dots,$$

$$t^{1-\gamma} u_{2,m}(t) = \int_0^1 t^{1-\gamma} U_2(t,y) \tilde{\omega}_{2,m}(y) dy \equiv 0, \quad m = 1, 2, \dots,$$

$$t^{1-\gamma} \tilde{u}_{2,m}(t) = \int_0^1 t^{1-\gamma} \tilde{U}_2(t,y) \omega_{2,m}(y) dy \equiv 0, \quad m = 1, 2, \dots$$

Hence, due to the completeness of systems of eigenfunctions $\{\omega_{1,n}(x)\}_{n=1}^\infty$, $\{\omega_{2,m}(x)\}_{m=1}^\infty$, $\{\tilde{\omega}_{2,m}(x)\}_{m=1}^\infty$ in space $L_2[0, 1]$, we conclude that $t^{1-\gamma}[U_0(t,x) + U_1(t,x) + U_2(t,x) + \tilde{U}_2(t,x)] \equiv 0$ for all $x \in [0, 1]$ and $t \in [0, T]$. Consequently, if mixed problem (1)–(5) has a solution, then this solution is unique in the domain $[0, 1]$.

Let us use the following well-known Banach space. Space $L_2[0, 1]$ of square summable functions on an interval $[0, 1]$ with a norm

$$\|\eta(x)\|_{L_2[0,1]} = \sqrt{\int_0^1 |\eta(y)|^2 dy} < \infty.$$

Smoothness condition. Let in the segment $[0, 1]$ the function $\varphi(x) \in C^5[0, 1]$ has the piecewise continuous derivative with respect to x up to the sixth order. Then, we integrate by parts

$$\varphi_{1,n} = \int_0^1 \varphi(y) \omega_{1,n}(y) dy, \quad \varphi_{2,m} = \int_0^1 \varphi(y) \tilde{\omega}_{2,m}(y) dy, \quad \tilde{\varphi}_{2,m} = \int_0^1 \varphi(y) \omega_{2,m}(y) dy$$

six times on the variable x and obtain the results

$$|\varphi_{1,n}| \leq \left(\frac{p+q}{2q\pi}\right)^4 \frac{|\varphi_{1,n}^{(IV)}|}{n^4}, \quad |\varphi_{2,m}| \leq \left(\frac{1}{2q\pi}\right)^4 \frac{|\varphi_{2,m}^{(IV)}|}{m^4} + 4 \left(\frac{1}{2q\pi}\right)^5 \frac{|\tilde{\varphi}_{2,m}^{(IV)}|}{m^5}, \quad (62)$$

$$\tilde{\varphi}_{2,m} \leq \left(\frac{1}{2q\pi}\right)^6 \frac{|\tilde{\varphi}_{2,m}^{(VI)}|}{m^6},$$

where

$$\varphi_{1,n}^{(IV)} = \int_0^1 \frac{\partial^4 \varphi(y)}{\partial y^4} \omega_{1,n}(y) dy, \quad \varphi_{2,m}^{(IV)} = \int_0^1 \frac{\partial^4 \varphi(y)}{\partial y^4} \tilde{\omega}_{2,m}(y) dy,$$

$$\tilde{\varphi}_{2,m}^{(VI)} = \int_0^1 \frac{\partial^6 \varphi(y)}{\partial y^6} \omega_{2,m}(y) dy.$$

We will use the following property of the Mittag-Leffler function [31, vol. 1, pp. 269–295]: for all $\alpha \in (0, 1)$, $\gamma \in \mathbb{R}$ and $\arg z = \pi$ there takes place the following estimate

$$|E_{\alpha,\gamma}(z)| \leq \frac{M_0}{1+|z|},$$

where $0 < M_0 = \text{const}$ does not depend from z . In particularly, for all $0 < \alpha \leq \gamma \leq 1$, $0 < \mu_{i,n} < 1$, $i = 1, 2$, we have the estimate

$$|E_{\alpha,\gamma}(-\mu_{i,n} t^\alpha)| \leq M_0.$$

Theorem. Let the condition A and smoothness condition be satisfied. Then for the regular values of the parameter ν the functions (36), (49), (51) and (61) converge absolutely and uniformly in the domain Ω . Moreover, the solution of the mixed problem (1)–(4) belongs to the class of functions (5).

Proof. It is not difficult to see that the function (36) is bounded. Let us show the absolute and uniform convergence of the Fourier series (49). The proof of convergence for other series (51) is similar. Taking (44) and (62) into account and applying the Cauchy–Schwartz inequality, for the regular values of the parameter v from series (49) we derive

$$\begin{aligned} |t^{1-\gamma}U_1(t, x)| &\leq \sum_{n=1}^{\infty*} |\vartheta_{1,n}(x)| \left[|\varphi_{1,n}E_{\alpha,\gamma}(-\mu_{1,n}t^\alpha)| + \right. \\ &\quad \left. + \frac{|v|}{\lambda_{1,n}} \sum_{r=1}^k \frac{|\Delta_{1,r,n}(v)|}{|\Delta_{1,n}(v)|} t^{1-\gamma+r} \left| \frac{1}{r!} - E_{\alpha,r+1}(-\mu_{1,n}t^\alpha) \right| r! \right] \leq \\ &\leq M_0 \left(\frac{p+q}{2q\pi} \right)^4 \sum_{n=1}^{\infty*} \frac{|\varphi_{1,n}^{(IV)}|}{n^4} + C_2 M_1 \left(\frac{p+q}{2q\pi} \right)^2 \sum_{n=1}^{\infty*} \frac{1}{n^2} |\varphi_{1,n}| \leq \\ &\leq M_0 \left(\frac{p+q}{2q\pi} \right)^4 \sqrt{\sum_{n=1}^{\infty*} \frac{1}{n^8}} \left\| \frac{\partial^4 \varphi_1(x)}{\partial x^4} \right\|_{L_2[0,1]} + C_2 M_1 \left(\frac{p+q}{2q\pi} \right)^2 \sqrt{\sum_{n=1}^{\infty*} \frac{1}{n^4}} \|\varphi_1(x)\|_{L_2[0,1]} < \infty, \end{aligned}$$

where $0 < C_2 = \text{const}$,

$$M_1 = |v| \max_{0 \leq t \leq T} \max_n \sum_{r=1}^k \frac{t^{1-\gamma+r}}{|\Delta_{1,n}(v)|} \left| \frac{1}{r!} - E_{\alpha,r+1}(-\mu_{1,n}t^\alpha) \right| r! < \infty.$$

Taking representations (58), estimates (53) and (62) into account, for the regular values of the parameter v from series (61) we obtain

$$\begin{aligned} |t^{1-\gamma}U_2(t, x)| &\leq \sum_{m=1}^{\infty} |\vartheta_{2,m}(x)| \left[|\varphi_{2,m}E_{\alpha,\gamma}(-\mu_{2,m}t^\alpha)| + \right. \\ &\quad \left. + \int_0^t t^{1-\gamma} s^r (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\mu_{2,m}(t-s)^\alpha) |h_m(s)| ds + \right. \\ &\quad \left. + \frac{|v|}{\lambda_{2,m}} \sum_{r=1}^k \frac{|\Delta_{2,m,r}(v)|}{|\Delta_{2,m}(v)|} t^{1-\gamma+r} \left| \frac{1}{r!} - E_{\alpha,r+1}(-\mu_{2,m}t^\alpha) \right| r! \right] \leq \\ &\leq M_0 \sum_{m=1}^{\infty} |\varphi_{2,m}| + M_2 \sum_{m=1}^{\infty} \frac{1}{\lambda_{2,m}} |\varphi_{2,m}| + M_0 \max_t |h_m(t)| \int_0^t t^{1-\gamma} s^r (t-s)^{\alpha-1} ds \leq \\ &\leq M_0 \sum_{m=1}^{\infty} \left[\frac{|\varphi_{2,m}^{(IV)}|}{\lambda_{2,m}^2} + 4 \frac{|\tilde{\varphi}_{2,m}^{(VI)}|}{\lambda_{2,m}^2 \sqrt{\lambda_{2,m}}} \right] + \\ &\quad + M_2 \sqrt{\sum_{m=1}^{\infty} \frac{1}{\lambda_{2,m}^2}} \|\varphi_2(x)\|_{L_2[0,1]} + C_3 \sqrt{\sum_{m=1}^{\infty} \frac{1}{\lambda_{2,m}}} \|\tilde{\varphi}_2(x)\|_{L_2[0,1]} \leq \\ &\leq M_0 \sum_{m=1}^{\infty} \left[\sqrt{\sum_{m=1}^{\infty} \frac{1}{\lambda_{2,m}^4}} \left\| \frac{\partial^4 \varphi_2(x)}{\partial x^4} \right\|_{L_2[0,1]} + 4 \sqrt{\sum_{m=1}^{\infty} \frac{1}{\lambda_{2,m}^5}} \left\| \frac{\partial^6 \tilde{\varphi}_2(x)}{\partial x^6} \right\|_{L_2[0,1]} \right] + \\ &\quad + M_2 \sqrt{\sum_{m=1}^{\infty} \frac{1}{\lambda_{2,m}^2}} \|\varphi_2(x)\|_{L_2[0,1]} + C_3 \sqrt{\sum_{m=1}^{\infty} \frac{1}{\lambda_{2,m}}} \|\tilde{\varphi}_2(x)\|_{L_2[0,1]} < \infty, \end{aligned}$$

where

$$C_3 = C_2 M_0 \max_t \int_0^t t^{1-\gamma} s^r (t-s)^{\alpha-1} ds, \quad \lambda_{2,m} = (2qm\pi)^2,$$

$$M_2 = |v| \max_{0 \leq t \leq T} \max_m \sum_{r=1}^k \frac{t^{1-\gamma+r}}{|\Delta_{2,m}(v)|} \left| \frac{1}{r!} - E_{\alpha,r+1}(-\mu_{2,m} t^\alpha) \right| r! < \infty.$$

It is easy to see that

$$\max \left\{ |t^{1-\gamma} D^{\alpha,\gamma} U_i(t,x)|, |t^{1-\gamma} D^{\alpha,\gamma} U_{ixx}(t,x)| \right\} < \infty, \quad i = 1, 2,$$

$$\max \left\{ |t^{1-\gamma} D^{\alpha,\gamma} \tilde{U}_2(t,x)|, |t^{1-\gamma} D^{\alpha,\gamma} \tilde{U}_{2xx}(t,x)| \right\} < \infty.$$

The theorem is proved.

References

- [1] O. KH. ABDULLAEV, O. SH. SALMANOV, T. K. YULDASHEV, *Direct and inverse problems for a parabolic-hyperbolic equation involving Riemann–Liouville derivatives*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. – Mathematics Vol. 43, 1 (2023), 21–33.
- [2] O. KH. ABDULLAEV, T. K. YULDASHEV, *Inverse problems for the loaded parabolic-hyperbolic equation involves Riemann–Liouville operator*, Lobachevskii J. Math. Vol. 44, 3 (2023), 1080–1090. <https://doi.org/10.1134/S1995080223030034>.
- [3] E. ALVAREZ, A. GOMEZ, M. PINTO, (ω, c) -periodic functions and mild solutions to abstract fractional integro-differential equations, Electron. J. Qual. Theory Differ. Equ. Vol. 16 (2018), 1–8. <https://doi.org/10.14232/ejqtde.2018.1.16>
- [4] B. AHMAD, A. ALSAEDI, M. KIRANE, R. G. TAPDIGOGLU, *An inverse problem for space and time fractional evolution equations with an involution perturbation*, Quaestiones Mathematicae Vol. 40, 2 (2017), 151–160. <https://doi.org/10.2989/16073606.2017.1283370>
- [5] R. R. ASHUROV, O. T. MUHIDDINOVA, *Initial-boundary value problem for a time-fractional subdiffusion equation with an arbitrary elliptic differential operator*, Lobachevskii J. Math. Vol. 42, 3 (2021), 517–525.
- [6] R. R. ASHUROV, O. T. MUHIDDINOVA, S. R. UMAROV, *A Non-local problem for the fractional-order Rayleigh–Stokes equation*, Fractal and Fractional Vol. 7, 6, ID 490 (2023), 1–16. <https://doi.org/10.3390/fractalfract7060490>
- [7] R. R. ASHUROV, O. T. MUHIDDINOVA, *Inverse problem of determining the order of the fractional derivative in the Rayleigh–Stokes equation*, Fractional Calculus and Applied Analysis Vol. 26 (2023), 1691–1708. <https://doi.org/10.1007/s13540-023-00178-9>
- [8] R. R. ASHUROV, YU. E. FAYZIEV, *Uniqueness and existence for inverse problem of determining an order of time-fractional derivative of subdiffusion equation*, Lobachevskii J. Math. Vol. 42, 3 (2021), 508–516. <https://doi.org/10.1134/S1995080221030069>
- [9] R. R. ASHUROV, YU. E. FAYZIEV, N. KH. KHUSHVAKTOV, *Some problems for the Barenblatt–Zhel'tov–Kochina type fractional equations*, Bulletin of the Institute of Mathematics Vol. 5, 5 (2022), 97–104.

- [10] R. R. ASHUROV, YU. E. FAYZIEV, N. KH. KHUSHVAKTOV, *Forward and inverse problems for the Barenblatt–Zhel'tov–Kochina type fractional equations*, Lobachevskii J. Math. Vol. 44, 7 (2023), 2563–2572.
- [11] R. R. ASHUROV, M. D. SHAKAROVA, *Time-Dependent source identification problem for fractional Schrödinger type equations*, Lobachevskii J. Math. Vol. 43, 5 (2022), 1053–1064.
- [12] R. R. ASHUROV, N. Z. VAISOVA, *Backward and non-local problems for the Rayleigh–Stokes equation*, Fractal and Fractional Vol. 6, 587 (2022), 1–18. <https://doi.org/10.3390/fractalfract6100587>
- [13] E. ALVAREZ, S. DIAZ, C. LIZAMA, *On the existence and uniqueness of (ω, c) –periodic solutions to a class of Volterra difference equations*, Adv. Differ. Equ. Vol. 2019, 105 (2019), 1–12. <https://doi.org/10.1186/s13662-019-2053-0>
- [14] M. AGAOGLOU, M. FECKAN, A. P. PANAGIOTIDOU, *Existence and uniqueness of (ω, c) –periodic solutions of semilinear evolution equations*, Int. J. Dyn. Sys. Diff. Equ. Vol. 10, 2 (2020), 149–166. <https://doi.org/10.1504/IJDSDE.2020.106027>
- [15] T. AKRAM, M. ABBAS, A. ALI, *A numerical study on time fractional Fisher equation using an extended cubic B-spline approximation*, J. Math. Comput. Sci. Vol. 22, 1 (2021), 85–96. <http://dx.doi.org/10.22436/jmcs.022.01.08>
- [16] R. ALAHMAD, Q. ALAHMAD, A. ABDELHADI, *Solution of fractional autonomous ordinary differential equations*, J. Math. Comput. Sci. Vol. 27, 1 (2020), 59–64. <http://dx.doi.org/10.22436/jmcs.027.01.05>
- [17] R. ASHUROV, B. KADIRKULOV, M. JALILOV, *On an inverse problem of the Bitsadze–Samarskii type for a parabolic equation of fractional order*, Boletín de la Sociedad Matemática Mexicana Vol. 29, 3 (2023), 1–21. <https://doi.org/10.1007/s40590-023-00542-y>
- [18] A. YU. ANIKIN, S. YU. DOBROKHOTOV, A. A. SHKALIKOV, *On expansions in the exact and asymptotic eigenfunctions of the one-dimensional Schrödinger operator*, Math. Notes Vol. 112, 5 (2022), 623–641.
- [19] G. I. BARENBLATT, YU. P. ZHELTOV, I. N. KOCHINA, *On finiteness conditions in the mechanics of continuous media. Static problems of the theory of elasticity*, Prikl. Mat. i Mekh. Vol. 24, 5 (1960), 316–322.
- [20] V. D. BUDAEV, *Orthogonal and biorthogonal bases*, Proceedings of the Russian State Pedagogical University named by A. I. Herzen Vol. 5 (2005), 7–38. [in Russian]
- [21] A. T. BULABAEV, M. OTELBAEV, L. A. SHUSTER, *Properties of Green's function of the Sturm–Liouville operator and their applications*, Differ. Equations Vol. 25, 7 (1989), 773–779.
- [22] G. I. BARENBLATT, YU. P. ZHELTOV, *Fundamental equations of filtration of homogeneous liquids in fissured rocks*, Sov. Phys., Dokl., Vol. 132, 5, (1960) 522–525.
- [23] A. S. BERDYSHEV, J. B. KADIRKULOV, *On a nonlocal problem for a fourth-order parabolic equation with the fractional Dzhrbashyan–Nersesyan operator*, Differential Equations Vol. 52, 1 (2016), 122–127. <https://doi.org/10.1134/S0012266116010109>
- [24] S. A. BERDYSHEV, A. CABADA, J. B. KADIRKULOV, *The Samarskii–Ionkin type problem for the fourth order parabolic equation with fractional differential operator*, Computers and Mathematics with Applications Vol. 62 (2011), 3884–3893.

- [25] M. KH. BESHTOKOV, *To boundary-value problems for degenerating pseudoparabolic equations with Gerasimov–Caputo fractional derivative*, Russian Mathematics (Izv. vuz) Vol. 62, 10 (2018), 1–14.
- [26] G. I. BARENBLATT, V. M. YENTOV, V. M. RYZHIK, *Movement of Liquids and Gases in Natural Reservoirs*, Nedra, Moscow, 1984. [in Russian]
- [27] K. S. BASNIEV, I. N. KOCHINA, V. M. MAKSIMOV, *Underground Hydromechanics*, Nedra, Moscow, 1993. [in Russian]
- [28] P. N. DUC, H. D. BINH, L. D. LONG, H. T. KIM VAN, *Reconstructing the right-hand side of the Rayleigh–Stokes problem with nonlocal in time condition*, Advances in Difference Equations Vol. 470 (2021), 1–18. <https://doi.org/10.1186/s13662-021-03626-z>
- [29] J. T. EDWARDS, N. J. FORD, A. C. SIMPSON, *The numerical solution of linear multi-term fractional differential equations: systems of equations*, J. Computational and Appl. Math., Vol. 148, 2 (2002), 401–418. [https://doi.org/10.1016/S0377-0427\(02\)00558-7](https://doi.org/10.1016/S0377-0427(02)00558-7)
- [30] V. E. FEDOROV, A. V. NAGUMANOVA, *Inverse linear problems for a certain class of degenerate fractional evolution equations*, Itogi nauki i tekhniki. Seriya Sovremennaya matematika i ikh prilozheniya. Tematicheskiye obzory, Vol. 167 (2019), 97–111. [in Russian] <https://doi.org/10.36535/0233-6723-2019-167-97-111>
- [31] R. GORENFLO, A. A. KILBAS, F. MAINARDI, S. V. ROGOZIN, *Mittag–Leffler Functions, Related Topics and Applications*, Springer, Berlin - Heidelberg, Germany, 2014, <https://doi.org/10.1007/978-3-662-61550-8>.
- [32] V. A. IL'IN, *Necessary and sufficient conditions for being a Riesz basis of root vectors of second-order discontinuous operators*, Differ. Uravneniya Vol. 22, 12 (1986), 2059–2071. [in Russian]
- [33] H. K. JASSIM, M. SHAREEF, *On approximate solutions for fractional system of differential equations with Caputo–Fabrizio fractional operator*, J. Math. Comput. Sci. Vol. 23, 1 (2021), 58–66. <http://dx.doi.org/10.22436/jmcs.023.01.06>
- [34] P. YA. KOCHINA AND OTHER, *Development of Research on the Theory of Filtration in the USSR (1917–1967)* Nauka, Moscow, 1969 [in Russian]. 138–144.
- [35] A. A. KILBAS, H. M. SRIVASTAVA, J. J. TRUJILLO, *Theory and Applications of Fractional Differential Equations*, Elsevier, North-Holland, Mathematics studies, 2006.
- [36] D. KUMAR, D. BALEANU, *Fractional calculus and its applications in physics*, Front. Phys. Vol. 7, 6 (2019). <https://doi.org/10.3389/fphy.2019.00081>.
- [37] M-H. KIM, G-CH. RI, H-CH. O, *Operational method for solving multi-term fractional differential equations with the generalized fractional derivatives* Fract. Calc. Appl. Anal. Vol. 17, 1 (2014), 79–95. <https://doi.org/10.2478/s13540-014-0156-6>.
- [38] M. T. KHALLADI, A. RAHMANI, (ω, c) –pseudo almost periodic distributions, Nonauton. Dyn. Syst. No. 7 (2020), 237–248. <https://doi.org/10.1515/msds-2020-0119>
- [39] L. V. KRITSKOV, A. M. SARSENBI, *Spectral properties of a nonlocal problem for a second-order differential equation with an involution*, Differential Equations Vol. 51, 8 (2015), 984–990. <https://doi.org/10.1134/S0012266115080029>
- [40] L. V. KRITSKOV, M. A. SADYBEKOV, A. M. SARSENBI, *Properties in L_p of root functions for a nonlocal problem with involution*, Turkish J. Math. Vol. 43 (2019), 393–401.

- [41] S. KERBAL, B. J. KADIRKULOV, M. KIRANE, *Direct and inverse problems for a Samarskii–Ionkin type problem for a two dimensional fractional parabolic equation*, Progr. Fract. Differ. Appl. Vol. 4, 3 (2018), 1–14. <http://dx.doi.org/10.18576/pfda/01010>
- [42] C. LIZAMA, *Abstract Linear Fractional Evolution Equations*, Handbook of Fractional Calculus with Applications Vol. 2, J.A.T. Machado Ed. DeGruyter, Berlin, 2019. p. 465–497.
- [43] Y. LIU, ZH. LI, M. YAMAMOTO, *Inverse problems of determining sources of the fractional partial differential equations*, Handbook of Fractional Calculus with Applications Vol. 2, J.A.T. Machado Ed. DeGruyter, Berlin, 2019. p. 411–429.
- [44] N. K. OCHILOVA, T. K. YULDASHEV, *On a nonlocal boundary value problem for a degenerate parabolic-hyperbolic equation with fractional derivative*, Lobachevskii J. Math. Vol. 43, 1 (2022), 229–236. <https://doi.org/10.1134/S1995080222040175>
- [45] M. PAKDAMAN, A. AHMADIAN, S. EFFATI, S. SALAHSHOUR, D. BALEANU, *Solving differential equations of fractional order using an optimization technique based on training artificial neural network*, Appl. Math. Comput. Vol. 293 (2017), 81–95. <https://doi.org/10.1016/j.amc.2016.07.021>
- [46] A. V. PSKHU, *Fractional Partial Differential Equations*, Nauka, Moscow, 2005. [in Russian] <https://www.elibrary.ru/item.asp?id=19448655>
- [47] S. PATNAIK, J. P. HOLLKAMP, F. SEMPERLOTTI, *Applications of variable-order fractional operators: a review*, Proceedings A, Royal Society, A476: 20190498. <http://dx.doi.org/10.1098/rspa.2019.0498>
- [48] P. YA. POLUBARINOVA-KOCHINA, *The Theory of Groundwater Movement*, Nauka, Moscow, 1977. [in Russian]
- [49] M. A. SADYBEKOV, G. DILDABEK, A. TENGAYEVA, *Constructing a basis from systems of eigenfunctions of one not strengthened regular boundary value problem*, Filomat Vol. 31, 4 (2017), 981–987.
- [50] R. K. SAXENA, R. GARRA, E. ORSINGER, *Analytical solution of space-time fractional telegraph-type equations involving Hilfer and Hadamard derivatives*, Integral Transforms and Special Functions No. 6 (2015). <https://doi.org/10.1080/10652469.2015.1092142>.
- [51] G. A. SVIRIDYUK, D. E. SHAFRANOV, *The Cauchy problem for the Barenblatt–Zhel'tov–Kochina equation on a smooth manifold*, Vest. Chel'yabinskGU Vol. 9 (2003), 171–177.
- [52] M. A. SAGADEEVA, F. L. HASAN, *Bounded solutions of Barenblatt–Zhel'tov–Kochina model in Quasi-Sobolev spaces*, Bulletin of the South Ural State University. Ser. Mathematical Modelling, Programming and Computer Software Vol. 8, 4 (2015),
- [53] H. SUN, A. CHANG, Y. ZHANG, W. CHEN, *A review on variable-order fractional differential equations: mathematical foundations, physical models, numerical methods and applications*, Fract. Calc. Appl. Anal. Vol. 22 (2019), 27–59. <https://doi.org/10.1515/fca-2019-0003>
- [54] A. M. SAVCHUK, A. A. SHKALIKOV, *Recovering a potential of Sturm–Liouville problem from finite sets of spectral data*, American Mathematical Society Translations – Series 2, Advances in the Mathematical Sciences Vol. 233 (2014), 211–224.
- [55] A. A. SHKALIKOV, *The basis problem of the eigenfunctions of ordinary differential operators with integral boundary conditions*, Moscow Univ. Math. Bull. Vol. 37, 6 (1982), 10–20.
- [56] A. A. SHKALIKOV, *Basis properties of root functions of differential operators with spectral parameter in the boundary conditions*, Differen. Equations Vol. 55, 5 (2019), 631–643.

- [57] J. A. TENREIRO MACHADO *Handbook of Fractional Calculus with Applications in eight volumes*, Walter de Gruyter GmbH, Berlin - Boston, 2019, p. 47–85.
- [58] V. E. VLADYKINA, A. A. SHKALIKOV, *Spectral properties of ordinary differential operators with involution*, Doklady Mathematics Vol. 99, 1 (2019), 5–10.
- [59] T. K. YULDASHEV, O. KH. ABDULLAEV, *Unique solvability of a boundary value problem for a loaded fractional parabolic-hyperbolic equation with nonlinear terms*, Lobachevskii J. Math. Vol. 42, 5 (2021), 1113–1123. <https://doi.org/10.1134/S1995080221050218>
- [60] T. K. YULDASHEV, B. J. KADIRKULOV, *Nonlocal problem for a mixed type fourth-order differential equation with Hilfer fractional operator*, Ural Math. J. Vol. 6, 1 (2020), 153–167.
- [61] T. K. YULDASHEV, T. G. ERGASHEV, T. A. ABDUVAHOBOV, *Nonlinear system of impulsive integro-differential equations with Hilfer fractional operator and mixed maxima*, Chelyabinsk. Phys. Math. J. Vol. 7, 3 (2022), 312–325.
- [62] T. K. YULDASHEV, KH. KH. SABUROV, T. A. ABDUVAHOBOV, *Nonlocal problem for a nonlinear system of fractional order impulsive integro-differential equations with maxima*, Chelyab. Phys.-Math. J. Vol. 7, 1 (2022), 113–122.
- [63] T. K. YULDASHEV, T. A. ABDUVAHOBOV, *Periodic solutions for an impulsive system of fractional order integro-differential equations with maxima*, Lobachevskii J. Math. Vol. 44, 10 (2023), 4393–4401, <https://doi.org/10.1134/S1995080223100451>
- [64] T. K. YULDASHEV, KH. R. MAMEDOV, T. A. ABDUVAHOBOV, *On a periodic solution for an impulsive system of differential equations with Gerasimov–Caputo fractional operator and maxima*, J. Contemporary Applied Math. Vol. 13, 1 (2023), 111–122. <http://journalcam.com>
- [65] T. K. YULDASHEV, KH. R. MAMEDOV, T. A. ABDUVAHOBOV, *(ω, c) –periodic solution for an impulsive system of differential equations with the quadrate of Gerasimov–Caputo fractional operator and maxima*, J. Contemporary Appl. Math., Vol. 13, 2 (2023), 65–79, <http://doi.org/10.5281/zenodo.8425548>, <http://journalcam.com>
- [66] T. K. YULDASHEV, B. J. KADIRKULOV, *Boundary value problem for weak nonlinear partial differential equations of mixed type with fractional Hilfer operator*, Axioms Vol. 9, 2 (2020), 1–19.
- [67] T. K. YULDASHEV, B. J. KADIRKULOV, *Inverse boundary value problem for a fractional differential equations of mixed type with integral redefinition conditions*, Lobachevskii J. Math. Vol. 42, 3 (2021), 649–662.
- [68] T. K. YULDASHEV, B. J. KADIRKULOV, *Inverse problem for a partial differential equation with Gerasimov–Caputo-type operator and degeneration*, Fractal Fract Vol. 5, 2, ID 58 (2021), 1–13. <https://doi.org/10.3390/fractalfract5020058>
- [69] T. K. YULDASHEV, B. J. KADIRKULOV, *On a boundary value problem for a mixed type fractional differential equations with parameters*, Proceedings of the Institute of Mathematics and Mechanics of Azerbaijan National Academy of Sciences Vol. 47, 1 (2021), 112–123.
- [70] T. K. YULDASHEV, B. J. KADIRKULOV, R. A. BANDALIYEV, *On a mixed problem for Hilfer type fractional differential equation with degeneration*, Lobachevskii J. Math. Vol. 43, 1 (2022), 263–274. <https://doi.org/10.1134/S1995080222040229>
- [71] T. K. YULDASHEV, E. T. KARIMOV, *Inverse problem for a mixed type integro-differential equation with fractional order Caputo operators and spectral parameters*, Axioms Vol 9, 4, ID 121 (2020), 1–24. <https://doi.org/10.3390/axioms9040121>

- [72] Y. ZHANG, X. XU, *Inverse source problem for a fractional differential equations*, Inverse Prob. Vol. 27, 3 (2011), 31–42.