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A Note on a Rothberger-Type Selection Principle and Normalized Rapidly Varying Sequences

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Abstract: In this paper, it is proven that the Rothberger selection hypothesis $S_1(NR_{\infty,s}, NR_{\infty,s})$ is valid, where $NR_{\infty,s}$ is the class of normalized rapidly varying sequences in the sense of De Haan.

Keywords and phrases: normalized rapidly varying sequences, Rothberger, selection principle.

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1 Introduction

The classical Karamata theory of regular variation (see [1]) has a functional and sequence aspects. It is a very important theory in asymptotic analysis, and especially in the analysis of divergent processes. The basic objects of this theory are slowly varying functions and slowly varying sequences (see [5] and [6]). Let $f : [a, +\infty) \to (0, +\infty)$ be a measurable function for which

$$\lim_{x\to+\infty} \frac{f(\lambda x)}{f(x)} = 1, \tag{1}$$

holds for every $\lambda > 0$. The function f is slowly varying (in the classical Karamata sense) and we denote the set of all such functions by SV_φ.

Theorem 1. *(Representation theorem) A function $f \in SV_\varphi$ if and only if for $x \geq a$, it holds $f(x) = c(x) \cdot \exp\{\int_a^x h(u)\frac{du}{u}\}$, where $c(x)$, $x \geq a$, is a measurable function for which $c(x) \to c \in \mathbb{R}$ holds for $x \to +\infty$, and the $h(u)$, $u \geq a$ is a measurable function for which $h(u) \to 0$, for $u \to +\infty$.*

Remark 1. *This theorem is the so-called Karamata theorem on the integral representation of elements from the class SV_φ (see [6]). A particularly important subclass of the class*

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SV_φ (e.g. in the theory of differential equations- see [1]) is the class of normalized slowly varying functions. It is denoted by NSV_φ . In NSV_φ are all functions of the class SV_φ for which $c(x) = c \in \mathbb{R}$, for $x \leq a$, in the previous theorem.

We say that a function $f : [a,+\infty) \to (0,+\infty)$ is a Zygmund function (see [1]) if for $\alpha > 0$, the function $x^\alpha \cdot f(x)$ increases for $x \geq x'_\alpha \geq a$, and the function $x^{-\alpha} \cdot f(x)$ decreases for $x \geq x'_\alpha \geq a$. We denote the set of Zygmund functions by Z_φ. This class of functions is an important object in Fourier analysis.

Theorem 2 (Bojanić and Karamata). *Let $f \in Z_\varphi$ and let it be an absolutely continuous function. Then $f \in NSV_\varphi$. The opposite is also true.*

Let us again consider a function $f : [a,+\infty) \to (0,+\infty)$ that is measurable and for which

$$\lim_{x\to+\infty} \frac{f(\lambda x)}{f(x)} = +\infty \tag{2}$$

holds for $\lambda > 1$. Then we say that the function f is rapidly varying in the sense of De Haan and we denote the set of all such functions by $R_{\infty,\varphi}$ [2].

Classes $R_{\infty,\varphi}$ and SV_φ are directly conjugate via the generalized inverse [3].

We say that a function $f : [a,+\infty) \to (0,+\infty)$, $a > 0$, is a normalized rapidly varying function (see [4]) if it is measurable and if for $x \geq a$, it holds that $f(x) = c \cdot \exp\{\int_a^x h(u)\frac{du}{u}\}$, where $c \in \mathbb{R}$ and $h(u)$, for $u \geq a$, is a measurable function for which $h(u) \to +\infty$, for $u \to \infty$.

Let (c_n) be a sequence of positive numbers. Then it is rapidly varying (see [2]) if

$$\lim_{n\to\infty} \frac{c_{[\lambda n]}}{c_n} = +\infty \tag{3}$$

holds for every $\lambda > 0$. We denote the class of such sequences by $R_{\infty,s}$ and it is an important object in the theory of selection principles and the theory of infinite topological games (see [2]).

2 The Main Results

We say that a sequence $c = (c_n) \in R_{\infty,s}$ is a normalized rapidly varying sequence if, for $n \in \mathbb{N}$, $c_n = c \cdot \exp\{\sum_{k=1}^n \frac{\delta_k}{k}\}$, where $c \in \mathbb{R}$ and (δ_k) is a sequence for which $\delta_k \to +\infty$ holds, for $k \to +\infty$. We denote the set of such sequences by $NR_{\infty,s}$.

Theorem 3. *The class $NR_{\infty,s}$ is a proper subclass of class $R_{\infty,s}$.*

Proof. Let a sequence $x = (x_n)$ be an element of the class $NR_{\infty,s}$. Then there exists a sequence (δ_k) for which $\delta_k \to +\infty$, for $k \to +\infty$, and there exists a constant $c \in \mathbb{R}^+$ such that $x_n = c \cdot \exp\sum_{k=1}^{n} \frac{\delta_k}{k}$, for every $n \in \mathbb{N}$. Let $\lambda > 0$ and $M > 0$. Then

$$
\begin{aligned}
\liminf_{n\to+\infty} \frac{x_{[\lambda n]}}{x_n} &= \liminf_{n\to+\infty} \exp\left\{\sum_{k=n}^{[\lambda n]} \frac{\delta_k}{k}\right\} \\
&\geq \liminf_{n\to+\infty} \exp\left\{M \cdot \sum_{k=n}^{[\lambda n]} \frac{\delta_k}{k}\right\} \\
&\geq \liminf_{n\to+\infty} \exp\left\{M \cdot \frac{1}{[\lambda n]}([\lambda n] - n + 1)\right\} \\
&\geq \liminf_{n\to+\infty} \exp\left\{(1 - \frac{n}{[\lambda n]} + \frac{1}{[\lambda n]})\frac{M}{1}\right\} \\
&\geq \liminf_{n\to+\infty} \exp\left\{(1 - \frac{n}{\lambda n} + \frac{1}{\lambda n}) \cdot M\right\} \\
&\geq \liminf_{n\to+\infty} \exp\left\{M \cdot (1 - \frac{1}{\lambda} + \frac{1}{\lambda n})\right\} \\
&= \exp\left\{M \cdot (1 - \frac{1}{\lambda})\right\}.
\end{aligned}
$$

For $M \to \infty$ we have that $\lim_{n\to\infty} \frac{x_{[\lambda n]}}{x_n} = +\infty$, and thus $x = (x_n) \in R_{\infty,s}$ holds. □

Consider the sequence $x = (x_n)$ given by $x_n = (2 + \sin n) \cdot e^{\sqrt{n}}$, for $n \in \mathbb{N}$. The sequence x belongs to $R_{\infty,s} \setminus NR_{\infty,s}$. The sequence x is not increasing for sufficiently large n, which means that it does not belong to the class $NR_{\infty,s}$.

In the theory of selection principles, one of the most important principles is S_1 Rothberger's principle. Namely, let $\mathscr{A}$ and $\mathscr{B}$ be sets whose elements are subsets of an infinite set $\mathscr{X}$. Then [2] $S_1(\mathscr{A}, \mathscr{B})$ represents the selection hypothesis:

(R) for every sequence $(A_n)_{n\in\mathbb{N}}$ of elements from $\mathscr{A}$ there is a sequence $(b_n)_{n\in\mathbb{N}}$ such that for every $n \in \mathbb{N}$, $b_n \in A_n$ and the set $\{b_n : n \in \mathbb{N}\}$ is an element from $\mathscr{B}$.

In the paper [2] it was shown that hypothesis $S_1(R_{\infty,s}, R_{\infty,s})$ is correct and in the next theorem we will upgrade the results.

Theorem 4. *Selection hypothesis $S_1(NR_{\infty,s}, NR_{\infty,s})$ holds.*

Proof. Consider a set of normalized rapidly varying sequences (x_{mn}) for $n \in \mathbb{N}$. Then there exist sequences (c_n), where $c_n \in \mathbb{R}$, $n \in \mathbb{N}$, (δ_{kn}) for $n \in \mathbb{N}$ and $k \in \{1, 2, ..., m\}$, $m \in \mathbb{N}$ such that

$$x_{m1} = c_1 \cdot \exp\left\{\sum_{k=1}^{m} \frac{\delta_{k_1}}{k}\right\}, m \in \mathbb{N},$$

$$\vdots$$

$$x_{mn} = c_n \cdot \exp\left\{\sum_{k=1}^{m} \frac{\delta_{k_n}}{k}\right\}, m \in \mathbb{N},$$

$$\vdots$$

where, for each $n \in \mathbb{N}$, $\lim_{k\to+\infty} \delta_{kn} = +\infty$ holds. For each $n \in \mathbb{N}$ and $k \in \{1,2,3,...,m\}$ from the sequence (δ_{kn}) we choose an element $\delta^*_{kn} \geq \delta_{kn} + k$, for each $k \in \{1,2,3,...,m\}$ and $n \in \mathbb{N}$. Then we can notice the sequence (α_k), such that for each fixed $n \in \mathbb{N}$, $\log \alpha_k = \{\frac{\delta^*_{kn}}{k}\}$ holds for each $k \in \mathbb{N}$. That is, for the same k, $\alpha_k = \exp\{\frac{\delta^*_{kn}}{k}\}$ and fixed $n \in \mathbb{N}$. Therefore, for $n \in \mathbb{N}$, we observe the sequence $y_n = \alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n$. Then the sequence $(y_n) \in NR_{\infty,s}$ and, for each $n \in \mathbb{N}$, the element y_n is an element of the sequence x_{mn}, for $m \in \mathbb{N}$. Hence, the selection principle $S_1(NR_{\infty,s}, NR_{\infty,s})$ holds.

$\square$

3 Conclusions

The results obtained in Theorem 3 and Theorem 4 can be the basis for getting new results in the theory of selection principles, Ramsey's theory, and the theory of infinite topological games for two players. It is interesting to prove for class $NR_{\infty,s}$ the correctness of Kočinac's selection hypotheses $\alpha_i(NR_{\infty,s}, NR_{\infty,s})$ for $i = 2,3,4$.

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