SCIENTIFIC PUBLICATIONS OF THE STATE UNIVERSITY OF NOVI PAZAR
SERIES A: APPL. MATH. INFORM. AND MECH. Vol. 17, 2, 2025, 87–93

On e-Menger Spaces

Ljubiša D. R. Kočinac, Diana Dolićanin Đekić

Abstract: We study a version of the classical Menger covering property, called e-Menger property, by using covers by so-called e-covers. Relations with other Menger-type properties and game-theoretic and Ramsey-theoretic characterizations of this property are given.

Keywords and phrases: e-open set, e-Menger space, e-perfect mapping, e-Menger game, partition relation.

2020 MSC. Primary: 54D20; Secondary: 05C55, 54C10, 91A44.

1 Introduction

In this paper me use the usual topological terminology and notation as in the book [4]. X and Y denote topological spaces, $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ are the closure and interior of a subset A of a space X. If $f : X \to Y$ is a mapping, then $f^{\leftarrow}(B)$ denotes the preimage of a set $B \subset Y$. By $\mathbb{N}$ and $\mathbb{R}$ we denote the set of natural numbers and the set of real numbers, respectively.

Recently, in [1], the notions of e-open set and e-space were introduced and their role in the study of algebraic and topological properties of the rings of e-continuous functions are demonstrated. For some other applications of these sets see the paper [2]. An open subset U of a topological space X is called *extremely open*, shortly e-open, if the closure $\mathrm{Cl}(U)$ of U is open in X. Clearly, any clopen (= open and closed) set is e-open. The complement of an e-open set is called e-closed. A space is said to be an e-space if it has a base consisting of e-open sets.

Lemma 1. ([1]) *The union and intersection of finitely many e-open sets is e-open.*

From Lemma 1 we get

Lemma 2. *If $\mathscr{U} = \{U_1, U_2, \cdots U_n, \cdot\}$ is a countable e-open cover of a space X, then $\mathscr{V} = \{V_1, V_2, \cdots V_n, \cdots\}$, where $V_k = \bigcup_{i \leq k} U_i$ for every $k \in \mathbb{N}$, is also an e-open cover of X.*

Manuscript received February 16, 2026; accepted February 21, 2026

Ljubiša D. R. Kočinac (ORCID 0000-0002-4870-7908) is with the Faculty of Science and Mathematics, University of Niš, Niš, Serbia, lkocinac@gmail.com

Diana Dolićanin Đekić (ORCID 0000-0002-0414-834X) is with Faculty of Technical Sciences, University of Priština in Kosovska Mitrovica, Kosovska Mitrovica, Serbia, dolicanin_d@yahoo.com

https://doi.org/10.46793/SPSUNP2502.087K

In this paper we use e-open sets in selection principles theory. Let $\mathscr{A}$ and $\mathscr{B}$ be collections of sets. The Menger-type selection principle, denoted $\mathsf{S}_{\mathsf{fin}}(\mathscr{A},\mathscr{B})$, is the following statement:

> For every sequence $(A_n)_{n\in\mathbb{N}}$ of elements of $\mathscr{A}$ there is a sequence $(B_n)_{n\in\mathbb{N}}$ of finite sets such for every n, $B_n \subseteq A_n$, and $\bigcup_{n\in\mathbb{N}} B_n \in \mathscr{B}$.

When $\mathscr{A} = \mathscr{B} = \mathscr{O}$, the collection of all open covers of a topological space X, then we have the classical Menger covering property of X [9, 5]. For more information on selection principles the reader can consult the papers [6, 10].

Note that $\mathsf{S}_{\mathsf{fin}}$ is antimonotone in the first coordinate, and monotone in the second coordinate. Denote by $\mathscr{O}_e$ the collection of all e-open covers of a space X. Since $\mathscr{O}_e \subseteq \mathscr{O}$ we have the following diagram:

$$\begin{tikzcd}
\mathsf{S}_{\mathsf{fin}}(\mathscr{O},\mathscr{O}) \arrow[r] \arrow[dr] & \mathsf{S}_{\mathsf{fin}}(\mathscr{O}_e,\mathscr{O}) \\
\mathsf{S}_{\mathsf{fin}}(\mathscr{O},\mathscr{O}_e) \arrow[r] \arrow[u] & \mathsf{S}_{\mathsf{fin}}(\mathscr{O}_e,\mathscr{O}_e) \arrow[u]
\end{tikzcd}$$

The spaces satisfying $\mathsf{S}_{\mathsf{fin}}(\mathscr{O}_e,\mathscr{O}_e)$ are called e-Menger, i.e.

Definition 1. A space X is *e-Menger* if for every sequence $(\mathscr{U}_n)_{n\in\mathbb{N}}$ of e-open covers of X there are finite sets $\mathscr{V}_n \subset \mathscr{U}_n$, $n \in \mathbb{N}$, such that $\bigcup_{n\in\mathbb{N}} \mathscr{V}_n$ covers X.

A space X is *e-compact* [1] (*e-Lindelöf*) if every e-open cover of X has a finite (countable) subcover. Evidently, every e-compact space is e-Menger, every e-Menger space is e-Lindelöf, and every Menger space is e-Menger. It is also evident that every e-closed subspace of an e-Menger space is also e-Menger.

Example 1. The space $[0,\omega_1)$ of all countable ordinals is not e-Menger because it is not e-Lindelöf: the clopen cover, hence e-open cover, $\{[0,\alpha) : \alpha \in [2,\omega_1)\}$ contains no a countable subcover.

2 Basic properties of e-Menger spaces

In this section we discuss relations between Mengerness and e-Mengerness is some classes of spaces and behavior of e-Mengerness under some kinds of mappings.

Theorem 1. *An e-space X is e-Menger if and only if it is Menger.*

Proof. Let X be an e-Menger space and let $(\mathscr{U}_n)_{n\in\mathbb{N}}$ be a sequence of open covers of X. As X is an e-space every $\mathscr{U}_n$ can be replaced by a cover $\mathscr{V}_n$ consisting of e-open basic sets. So, every $\mathscr{V}_n$ is an e-open cover of X. Apply to the sequence $(\mathscr{V}_n)_{n\in\mathbb{N}}$ the fact that X is e-Menger and find for every n a finite subset $\mathscr{W}_n$ of $\mathscr{V}_n$ such that $\bigcup_{n\in\mathbb{N}} \mathscr{W}_n$ is an e-open cover of X. For every n let $\mathscr{H}_n = \{U_W \in \mathscr{U}_n : U_W \supset W, W \in \mathscr{W}_n\}$. Evidently, the sequence $(\mathscr{H}_n)_{n\in\mathbb{N}}$ witnesses for $(\mathscr{U}_n)_{n\in\mathbb{N}}$ that X is a Menger space. $\square$

Recall that a space X is *zero-dimensional* (in the sense of small inductive dimension) if it has a base consisting of clopen sets [4]. Every zero-dimensional space is an e-space; the converse need not be true [1, Example 2.8].

Corollary 1. *A zero-dimensional space X is e-Menger if and only if it is Menger.*

Example 2. The Sorgenfrey line $\mathbb{S}$ and the space $\mathbb{P}$ of irrational numbers (with the Euklidean topology inherited from the real line $\mathbb{R}$) are not e-Menger because they are zero-dimensional spaces which are not Menger as it is well known.

We discuss now the behaviour of the e-Menger property under some classes of mappings.

Following [1] call a mapping $f : X \to Y$ is *e-continuous at $x \in X$* if for every open set $V \subset Y$ there is an e-open neighbourhoodn V of x such that $f(U) \subset V$. f is e-continuous if it is e-continuous at every $x \in X$.

Theorem 2. *If $f : X \to Y$ is e-continuous mapping from an e-Menger space X onto a space Y, then Y is e-Menger.*

Proof. Let $(\mathscr{V}_n)_{n\in\mathbb{N}}$ be a sequence of open covers of Y. For every $x \in X$ and every $n \in \mathbb{N}$ there is $V_{n,x} \in \mathscr{V}_n$ containing $f(x)$. Since f is e-continuous there is an e-open neighbourhood $U_{n,x}$ of x such that $f(U_{n,x}) \subset V_{n,x}$. Let $\mathscr{U}_n = \{U_{n,x} : x \in X\}$, $n \in \mathbb{N}$. Then $(\mathscr{U}_n)_{n\in\mathbb{N}}$ is a sequence of e-open covers of X. Since X is e-Menger, there is a sequence $(\mathscr{H}_n)_{n\in\mathbb{N}}$ such that for every n, $\mathscr{H}_n$ is a finite subset of $\mathscr{U}_n$ and $X = \bigcup_{n\in\mathbb{N}} \bigcup \mathscr{H}_n$. Set $\mathscr{W}_n = \{V_{n,x} : f(H) \subset V_{n,x}), H \in \mathscr{H}_n\}$. We get a sequence $(\mathscr{W}_n)_{n\in\mathbb{N}}$ of finite sets such that $\mathscr{W}_n \subset \mathscr{V}_n$ for every $n \in \mathbb{N}$. Thus $Y = \bigcup_{n\in\mathbb{N}} \bigcup_{W\in\mathscr{W}_n} W$, i.e. Y is e-Menger. □

Recall that a mapping $f : X \to Y$ is *contra-continuous* [3] if the preimage $f^{\leftarrow}(V)$ of an open set $V \subset Y$ is closed in X, and *precontinuous* [8] if $f^{\leftarrow}(V) \subset \mathrm{Int}(\mathrm{Cl}(f^{\leftarrow}(V)))$ whenever V is open in Y.

Theorem 3. *A contra-continuous and precontinuous image $Y = f(X)$ of an e-Menger space X is a Menger space.*

Proof. Let $(\mathscr{V}_n)_{n\in\mathbb{N}}$ be a sequence of open covers of Y. Since f is contra-continuous for every $n \in \mathbb{N}$ and every $V \in \mathscr{V}_n$ the set $f^{\leftarrow}(V)$ is closed in X. On the other hand, because f is precontinuous, $f^{\leftarrow}(V) \subset \mathrm{Int}(\mathrm{Cl}(f^{\leftarrow}(V)))$, so that $f^{\leftarrow}(V) \subset \mathrm{Int}(f^{\leftarrow}(V))$, i.e. $f^{\leftarrow}(V) = \mathrm{Int}(f^{\leftarrow}(V))$. Therefore, for every n, the set $\mathscr{U}_n = \{f^{\leftarrow}(V) : V \in \mathscr{V}_n\}$ is an e-cover of X, being a clopen cover of X. As X is an e-Menger space there is a sequence $(\mathscr{G}_n)_{n\in\mathbb{N}}$ such that for every n, $\mathscr{G}_n$ is a finite subset of $\mathscr{U}_n$ and $X = \bigcup_{n\in\mathbb{N}} \bigcup \mathscr{G}_n$. Let $\mathscr{W}_n = \{f(G) : G \in \mathscr{G}_n\}$. Then for every n, $\mathscr{W}_n$ is a finite subset of $\mathscr{V}_n$ and $Y = \bigcup_{n\in\mathbb{N}} \bigcup \mathscr{W}_n$. This means that Y is a Menger space. □

A mapping $f : X \to Y$ is called *weakly e-continuous* if for every $x \in X$ and every open neighbourhood V of $f(x)$ there is an e-open neighbourhood U of x such that $f(U) \subset \mathrm{Cl}(V)$. f is weakly e-continuous if it is weakly e-continuous at every point in X.

Recall that a space X is *almost Menger* if for every sequence $(\mathscr{U}_n)_{n\in\mathbb{N}}$ of open covers of X there is a sequence $(\mathscr{V}_n)_{n\in\mathbb{N}}$ of finite sets such that for every n, $\mathscr{V}_n \subseteq \mathscr{U}_n$, and $X = \bigcup_{n\in\mathbb{N}}\bigcup_{V\in\mathscr{V}_n}\mathrm{Cl}(V)$ [7, 6].

Theorem 4. *If $f : X \to Y$ is weakly e-continuous mapping from an e-Menger space X onto a space Y, then Y is almost Menger.*

Proof. Let $(\mathscr{V}_n)_{n\in\mathbb{N}}$ be a sequence of open covers of Y. For every $x \in X$ and every $n \in \mathbb{N}$ there is $V_{n,x} \in \mathscr{V}_n$ containing $f(x)$. Since f is weakly e-continuous there is an e-open neighbourhood $U_{n,x}$ of x such that $f(U_{n,x}) \subset \mathrm{Cl}(V_{n,x})$. Put $\mathscr{U}_n = \{U_{n,x} : x \in X\}$, $n \in \mathbb{N}$. Then $(\mathscr{U}_n)_{n\in\mathbb{N}}$ is a sequence of e-open covers of X. Since X is e-Menger, there is a sequence $(\mathscr{H}_n)_{n\in\mathbb{N}}$ such that for every n, $\mathscr{H}_n$ is a finite subset of $\mathscr{U}_n$ and $X = \bigcup_{n\in\mathbb{N}}\bigcup\mathscr{H}_n$. Set $\mathscr{W}_n = \{V_{n,x} : f(H) \subset \mathrm{Cl}(V_{n,x}), H \in \mathscr{H}_n\}$. We get a sequence $(\mathscr{W}_n)_{n\in\mathbb{N}}$ of finite sets such that $\mathscr{W}_n \subset \mathscr{V}_n$ for every $n \in \mathbb{N}$. Thus $Y = \bigcup_{n\in\mathbb{N}}\bigcup_{W\in\mathscr{W}_n}\mathrm{Cl}(W)$, i.e. Y is almost Menger. □

A mapping $f : X \to Y$ is

- *e-open* if the set $f(E)$ is e-open in Y for every e-open set $E \subset X$;
- *e-closed* if for every $y \in Y$ and every e-open set $O \supset f^{\leftarrow}(y)$ in X there is an e-open set $E \subset Y$ containing y such that $f^{\leftarrow}(E) \subset O$;
- *e-perfect* if it is e-closed and for every $y \in Y$ the set $f^{\leftarrow}(y)$ is e-compact in X.

Theorem 5. *If $f : X \to Y$ is an e-open, e-perfect mapping from a space X onto a Menger space Y, then X is e-Menger.*

Proof. Let $(\mathscr{U}_n)_{n\in\mathbb{N}}$ be a sequence of e-open covers of X. For every $y \in Y$ the set $f^{\leftarrow}(y)$ is e-compact so that for every $n \in \mathbb{N}$ there is a finite set $\mathscr{U}_{n,y} \subset \mathscr{U}_n$ covering $f^{\leftarrow}(y)$. Set $V_{n,y} = \bigcup\mathscr{U}_{n,y}$. As, by Lemma 1, $V_{n,y}$ is an e-open neighbourhood of $f^{\leftarrow}(y)$ and f is an e-closed mapping, for every $n \in \mathbb{N}$ and every $y \in Y$ there is an e-open set $W_{n,y} \subset Y$ such that $y \in W_{n,y}$ and $f^{\leftarrow}(W_{n,y}) \subset V_{n,y}$. For every $n \in \mathbb{N}$ set $\mathscr{W}_n = \{W_{n,y} : y \in Y\}$. Then every $\mathscr{W}_n$ is an e-open, hence open, cover of Y. Since Y is Menger, there is a sequence $(\mathscr{G}_n)_{n\in\mathbb{N}}$ such that $\mathscr{G}_n$ is a finite subset of $\mathscr{W}_n$, $n \in \mathbb{N}$, and $Y = \bigcup_{n\in\mathbb{N}}\bigcup\mathscr{G}_n$. For every n and every $G \in \mathscr{G}_n$ there is a finite $\mathscr{U}_{n,G} \subset \mathscr{U}_n$ with $f^{\leftarrow}(G) \subset \bigcup\mathscr{U}_{n,G}$. If $\mathscr{H}_n = \{U \in \mathscr{U}_n : U \in \mathscr{U}_{m,G}, G \in \mathscr{G}_n\}$, then $\mathscr{H}_n$ is a finite subset of $\mathscr{U}_n$ for every n. Then the sequence $(\mathscr{H}_n)_{n\in\mathbb{N}}$ witnesses for the given sequence $(\mathscr{U}_n)_{n\in\mathbb{N}}$ that X is e-Menger. □

3 Characterizations of e-Menger spaces

In this section we characterize the e-Menger property game-theoretically and Ramsey-theoretically.

The symbol $\mathsf{M_e}(X)$ denotes the following *e-Menger game* on X: players I and II play a round for every $n \in \mathbb{N}$. In the nth round player I chooses an e-open cover $\mathscr{U}_n$ of X, and then II chooses a finite set $\mathscr{V}_n \subset \mathscr{U}_n$. II wins a play $\mathscr{U}_1, \mathscr{V}_1; \mathscr{U}_2, \mathscr{V}_2; \cdots$ if $\bigcup_{n\in\mathbb{N}} \mathscr{V}_n$ covers X; otherwise I wins.

Note that a space X has the e-Menger property whenever I does not have a winning strategy in the game $\mathsf{M_e}(X)$.

Theorem 6. *For an e-Lindelöf space X the following statements are equivalent:*

(1) *X is an e-Menger space;*

(2) I *does not have a winning strategy in the game $M_e(X)$ on X.*

Proof. By the above observation we have to prove only $(1) \Rightarrow (2)$. Let σ be a strategy for I.

Round 1: In the first round I chooses a countable cover $\sigma(\emptyset) = \mathscr{U}_1 \in \mathscr{O}_e$, say $\mathscr{U}_1 = \{U_n : n \in \mathbb{N}\}$. Applying Lemma 2, I can consider the cover $\mathscr{V}_1 = \{V_1 \subset V_2 \subset \cdots \subset V_n \subset \cdots\}$ instead of $\mathscr{U}_1$ [In the next rounds I selects e-open covers $\mathscr{V}_2, \mathscr{V}_3, \cdots$ listed in the increasing order with respect to inclusion.] Suppose that II's response is V_{n_1} (which is actually the choice of the initial part $\{U_1, U_2, \cdots, U_{n_1}\}$ of $\mathscr{U}_1$).

Round 2: I chooses a countable e-open cover $\mathscr{V}_2 = \sigma(V_{n_1}) = \{V_{n_1,n} : n \in \mathbb{N}\}$. Suppose II's response is $V_{n_1,n_2} \supset V_{n_1}$.

Round 3: I selects an e-open cover $\mathscr{V}_3 = \sigma(V_{n_1}, V_{n_1,n_2}) = \{V_{n_1,n_2,n} : n \in \mathbb{N}\}$, and II chooses $V_{n_1,n_2,n_3} \supset V_{n_1,n_2}$.

And so on. In this way we get two sequences

$$\mathscr{V}_1, \mathscr{V}_2, \mathscr{V}_3, \cdots, \mathscr{V}_k, \cdots \quad \text{and} \quad V_{n_1} \subset V_{n_1,n_2} \subset \cdots \subset V_{n_1,n_2,\cdots,n_k} \supset \cdots.$$

For every $n, m \in \mathbb{N}$ define sets $W_{n,m}$ as follows:

(i) $W_{1,m} = V_m$;

(ii) If $n \geq 2$, $W_{n,m} = \bigcap\{V_{s\frown m} : s$ is a sequence of lenght $n-1\} \bigcap V_{n-1,m}$.

By Lemma 1 every set $W_{n,m}$ is e-open so that for every n, $\mathscr{W}_n = \{W_{n,m} : m \in \mathbb{N}\}$ is an e-open cover of X listed in an increasing order. As X is e-Menger there is a mapping $\varphi : \mathbb{N} \to \mathbb{N}$ such that $\{W_{n,\varphi(n)} : n \in \mathbb{N}\}$ covers X. Observe that $W_{n,\varphi(n)} \subset V_{\varphi\restriction n+1}$. It follows that the set $\{V_{\varphi(1)}, V_{\varphi(1),\varphi(2)}, V_{\varphi(1),\varphi(2),\varphi(3)}, \cdots, V_{\varphi(1),\varphi(2),\cdots,V_{\varphi(n)}}, \cdots\}$ of moves of II covers X, i.e. σ is not a winning strategy of I. $\square$

In the sequel we use the following notation for a space X:

- Ω_e denotes the collection of all e-open covers $\mathscr{U}$ of X such that every finite subset of X is contained in a member of $\mathscr{U}$ and $X \notin \mathscr{U}$. Elements in Ω_e are called *ω-e-covers*.

Let us note that any $\mathscr{U} \in \Omega_\mathrm{e}$ satisfies:

> For every k and every partition $\mathscr{U} = \mathscr{U}_1 \cup \cdots \mathscr{U}_k$ there is an $i \leq k$ belonging to Ω_e.

A space X is called *ω-e-Lindelöf* if every cover in Ω_e has a countable subcover.

Proposition 1. *For an e-Lindelöf space X the following statements are equivalent:*

(1) *X is an e-Menger space;*

(2) *X satisfies the selection principle $\mathsf{S}_{\mathsf{fin}}(\Omega_e, \mathscr{O}_e)$.*

Proof. Because $\Omega_e \subset \mathscr{O}_e$ we have $\mathsf{S}_{\mathsf{fin}}(\Omega_e, \mathscr{O}_e) \supset \mathsf{S}_{\mathsf{fin}}(\mathscr{O}_e, \mathscr{O}_e)$, i.e., $(1) \Rightarrow (2)$.

$(2) \Rightarrow (1)$ Let $(\mathscr{U}_n)_{n\in\mathbb{N}}$ be a sequence of e-open covers of X. Consider now the sequence $(\mathscr{V}_n)_{n\in\mathbb{N}}$ of e-open covers such that every $\mathscr{V}_n$ is the set of finite unions of elements of $\mathscr{U}_n$. In fact, every $\mathscr{V}_n$ is in Ω_e and is countable. By (2) there is a sequence $(\mathscr{W}_n)_{n\in\mathbb{N}}$ such that for every n, $\mathscr{W}_n$ ia a finite subset of $\mathscr{V}_n$ and $\bigcup_{n\in\mathbb{N}} \mathscr{W}_n \in \mathscr{O}_e$. But, every $W \in \mathscr{W}_n$ is a union of finitely many elements of $\mathscr{U}_n$. Therefore, disassembling elements of every $\mathscr{W}_n$ one gets a sequence $(\mathscr{H}_n)_{n\in\mathbb{N}}$ of finite subsets of the corresponding $\mathscr{U}_n$ witnessing that X is e-Menger. □

Recall the following concept in Ramsey theory, called the *Baumgartner-Taylor partition relation* (see [6, 10]). For every positive integer k,

$$\mathscr{A} \to \lceil \mathscr{B} \rceil_k^2$$

denotes the following:

> For every A in $\mathscr{A}$ and for every function $f : [A]^2 \to \{1, \cdots, k\}$ there are a set $B \in \mathscr{B}$ with $B \subset A$, a $j \in \{1, \cdots, k\}$, and a partition $B = \bigcup_{n\in\mathbb{N}} B_n$ of B into pairwise disjoint finite sets such that for every $\{p, q\} \in [B]^2$ for which p and q are not from the same B_n, $f(\{p, q\}) = j$ holds.

Here $[A]^2$ denotes the set of all two-element subsets of A. The set B is called nearly homogeneous of color j.

Theorem 7. *Let X be an e-Lindelöf space. Consider the following statements:*

(1) *The partition relation $\Omega_\mathrm{e} \to \lceil \mathscr{O}_e \rceil_2^2$ holds;*

(2) *X is an e-Menger space.*

Then $(1) \Rightarrow (2)$.

Proof. By Proposition 1 it is enough to prove that (1) implies the statement (2) in that proposition.

Let $(\mathscr{U}_n)_{n\in\mathbb{N}}$ be a sequence of countable elements of Ω_e, say $\mathscr{U}_n = \{U_{n,m} : m \in \mathbb{N}\}$ for every n. Let $\mathscr{V}$ be the collection of nonempty sets of the form $U_{1,n} \cap U_{n,m}$. It is clear that $\mathscr{V} \in \Omega_\mathrm{e}$. For every $V \in \mathscr{V}$ choose a representation of the form $V = U_{1,n} \cap U_{n,m}$. Then define the function $f : [\mathscr{V}]^2 \to \{1, 2\}$ as follows:

$$f(\{U_{1,n_1} \cap U_{n_1,i}, U_{1,n_2} \cap U_{n_2,j}\}) = \begin{cases} 1 & \text{if } n_1 = n_2, \\ 2 & \text{otherwise.} \end{cases}$$

By (1), take $\mathscr{W} \subset \mathscr{V}$ with $\mathscr{W} \in \mathscr{O}_{\mathrm{e}}$ and $\mathscr{W}$ is nearly homogeneous of color j. Assume $\mathscr{W} = \bigcup_{k\in\mathbb{N}} \mathscr{W}_k$ is a sequence of finite, pairwise disjoint sets such that for A and B from distinct $\mathscr{W}_k$'s we have $f(\{A,B\}) = j$. We have the following two possibilities.

Case 1. $j = 1$.

There is an n such that for all $A \in \mathscr{W}$ we have $A \subset U_{1,n}$. This implies that $\mathscr{W}$ does not belong to $\mathscr{O}_{\mathrm{e}}$. Thus, this case does not hold.

Case 2. $j = 2$.

For every $n > 1$ define

$$\mathscr{H}_n = \{U_{n,i} : U_{n,i} \text{ is the second coordinate in the representation of an } W \in \mathscr{W}\}$$

and set $\mathscr{H} = \bigcup_{n\in\mathbb{N}} \mathscr{H}_n$. Then $\mathscr{H}$ is the union of finite subsets of $\mathscr{U}_n$, $n \in \mathbb{N}$.

It is easily checked that $\mathscr{H} \in \mathscr{O}_{\mathrm{e}}$. If for some n, $\mathscr{H}_n$ was not defined, we set $\mathscr{H}_n = \emptyset$. So we get the sequence $(\mathscr{H}_n)_{n\in\mathbb{N}}$ witnessing for $(\mathscr{U}_n)_{n\in\mathbb{N}}$ that $\mathsf{S}_{\mathsf{fin}}(\Omega_e, \mathscr{O}_e)$ holds. $\square$

4 Conclusions

In this note, by employing e-open sets, we established several properties of the e-Menger covering property which is a generalization of the classical Menger covering property. Further work may be done for Rothberger-type and Hurewicz-type covering properties which involve e-open covers.

References

[1] S. Afrooz, F. Azarpanah, N. Hasan Hajee, *On e-spaces and rings of real valued e-continuous functions*, Appl. Gen. Topol. Vol. 24, 2 (2023), 433–448.

[2] R. B. Beshimov, R. M. Juraev, R. Z. Manasipova, *On τ-base and e-density of topological spaces*, Filomat Vol. 39, 10 (2025), 3353–3358.

[3] J. Dontchev, *Contra-continuous functions and strongly S-closed spaces*, Intern. J. Math. Math. Sci. Vol. 19 (1996), 303 - 310.

[4] R. Engelking, *General Topology*, 2nd ed., Sigma Ser. Pure Math., Vol. 6, Heldermann, Berlin, 1989.

[5] W. Hurewicz, *Über eine Verallgemeinerung des Borelschen Theorems*, Math. Z. Vol. 24 (1925), 401–421.

[6] Lj. D. R. Kočinac, *Selected results on selection principles*, In: Proceedings of the Third Seminar on Geometry and Topology, July 15–17, 2004, Tabriz, Iran, pp. 71–104.

[7] Lj. D. R. Kočinac, *Star-Menger and related spaces* II, Filomat Vol. 13 (1999), 129–140.

[8] A. S. Mashhour, M. E. Abd El-Monsef, S. N. El-Deep, *On precontinuous and weak precontinuous mappings*, Proc. Math. Phys. Soc. Egypt Vol. 53 (1982), 47–53.

[9] K. Menger, *Einige Überdeckungssätze der Punktmengenlehre*, Sitzungsberischte Abt. 2a, Math. Astron. Physik Meteorol. Mech. (Wiener Akademie, Wien) Vol. 133 (1924), 421- 444.

[10] M. Sakai, M. Scheepers, *The combinatorics of open covers,* In: K. P. Hart, J. van Mill, P. Simon (Eds.), *Recent Progress in General Topology III*, Atlantis Press (2014), 751–799.