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Extremal Degree-Ratio Sombor Index Trees

Ivan Gutman

Abstract: The Sombor index is a geometry-motivated vertex-degree-based graph invariant, defined as the sum of the terms $\sqrt{d(u)^2+d(v)^2}$ over all pairs of adjacent vertices; $d(u)$ is the degree of the vertex u. Recently, Nyauli and Buragohain, introduced a novel, "degree-ratio" variant of the Sombor index, defined via $\sqrt{[d(u)/d(v)]^2+[d(v)/d(u)]^2}$. In this paper, we determine the trees with maximum, second- and third-maximum, as well as minimum and second-minimum degree-ratio Sombor index.

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1 Introduction

The Sombor index, defined as

$$SO = SO(G) = \sum_{uv\in\mathbf{E}(G)} \sqrt{d(u)^2+d(v)^2} \tag{1}$$

is a vertex-degree-based graph invariant, conceived on geometry-based considerations [2]. Since its discovery in 2021, it attracted much attention and was extensively studied, see the reviews [3, 4, 7]. Quite a few variants of SO were put forward in the mathematical and mathematico-chemical literature [3]. One of the most recent such variant is the *degree-ratio Sombor index*, defined as [6]

$$DRSO = DRSO(G) = \sum_{uv\in\mathbf{E}(G)} \sqrt{\left[\frac{d(u)}{d(v)}\right]^2+\left[\frac{d(v)}{d(u)}\right]^2}\,. \tag{2}$$

The basic mathematical properties of $DRSO$ were established by Nyauli and Buragohain [6], who also indicated its geometric origin. The present paper is aimed as a further contribution to the emerging theory of degree-ratio Sombor index, by characterizing trees extremal w.r.t. $DRSO$.

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Ivan Gutman (ORCID 0000-0001-9681-1550) is with the Faculty of Science, University of Kragujevac, Kragujevac, Serbia, gutman@kg.ac.rs
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We use the following notation and terminology.

In Eqs. (1) and (2), G stands for a simple graph, with vertex set $\mathbf{V}(G)$ and edge set $\mathbf{E}(G)$. Then $n = |\mathbf{V}(G)|$ and $m = |\mathbf{E}(G)|$ are, respectively, the order and size of G, that is the number of its vertices and edges.

By $uv \in \mathbf{E}(G)$ we denote the edge of G connecting the vertices u and v. The degree of a vertex $u \in \mathbf{V}(G)$ (= number of vertices that are adjacent to u) is denoted by $d(u)$. In simple graphs, $d(u)$ is integer-valued. If u is the end-vertex of an edge of a graph of order n, then $1 \leq d(u) \leq n-1$.

An edge whose end-vertices have degrees i and j will be referred to as an (i,j)-type edge. If $j = 1$, then the respective edge is said to be pendent.

For other graph-theoretical notions, the readers are referred to textbooks [1, 5].

2 Preparations

From the definition of the degree-ratio Sombor index, Eq. (2), it is evident that the term

$$\gamma(uv) = \sqrt{\left[\frac{d(u)}{d(v)}\right]^2 + \left[\frac{d(v)}{d(u)}\right]^2}$$

is the contribution of the edge uv to the total value of $DRSO$. Thus, if uv is an (i,j)-type edge, then

$$\gamma(uv) = \gamma(i,j) = \sqrt{\left(\frac{i}{j}\right)^2 + \left(\frac{j}{i}\right)^2}$$

and then

$$DRSO(G) = \sum_{1 \leq j \leq i \leq n-1} m_{ij}\, \gamma(i,j) \tag{3}$$

where m_{ij} is the number of (i,j)-type edges of the graph G.

In what follows, without loss of generality, it will be assumed that $d(u) \geq d(v)$, i.e., $i \geq j$.

Bearing in mind the algebraic form of $DRSO$, Eq. (2), we first consider the function

$$f(x) = \sqrt{x^2 + \frac{1}{x^2}}.$$

By elementary calculus it can be shown that $f(x)$ monotonically decreases in the interval $(0,1)$, attains its minimum at $x = 1$, and monotonically increases for $x \in (1,\infty)$. In view of Eq. (2), we are interested in the function $f(x)$ in which $1 \leq x \leq n-1$. If so, then we immediately arrive at the following:

Lemma 1.
(a) $\gamma(i,j)$ *attains its minimal value if and only if* $i = j$. *Then* $\gamma(i,i) = \sqrt{2}$ *for any* $i \geq 1$.
(b) $\gamma(i,j)$ *attains its maximal value if and only if* $i = n-1$ *and* $j = 1$. *Then*

$$\gamma(n-1,1) = \sqrt{(n-1)^2 + (n-1)^{-2}}\,.$$

(c) *Among pendent edges (for which $j = 1$), $\gamma(2,1)$ has the smallest value, and $\gamma(i,1)$ monotonically increases with the parameter i.*

Remark 1. *Among graphs of size m, the minimal value of degree-ratio Sombor index is $\sqrt{2}m$. This value is attained by the graphs consisting of components, each of which is a regular graph (not necessarily of the same degree).*

We are now prepared to characterize trees with extremal *DRSO*-values.

3 Trees with Maximal Degree-Ratio Sombor Index

Let $\mathscr{T}_n$ be the set of all trees of order n. As usual, the n-vertex star and path are denoted by S_n and P_n.

Let $T_1, T_2, T_3, T_4, T_5 \in \mathscr{T}_n$ be trees whose 9-vertex representatives **1, 2, 3, 4, 5** are depicted in Figure 1. Note that $T_1 \cong S_n$.

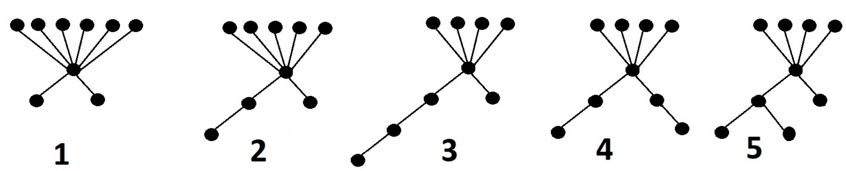

Figure 1. Trees of order 9 with greatest degree-ratio Sombor index.

Theorem 1. *The star $S_n \cong T_1$ is the n-vertex tree with greatest degree-ratio Sombor index.*

Proof. All edges of T_1 are of $(n-1,1)$-type. Noting this, Theorem 1 follows from Lemma 1 (b). □

The tree $T_2 \in \mathscr{T}_n$ is obtained by attaching a new pendent edge to a pendent edge of the star S_{n-1}; cf. graph **2** in Figure 1.

Theorem 2. *T_2 is the n-vertex tree with second-greatest degree-ratio Sombor index.*

Proof. The star S_n is the tree with the greatest number of pendent edges attached to the central vertex. The tree T_2 is the unique element of $\mathscr{T}_n$ possessing the second-greatest number of such pendent edges (equal to $n-3$). Again, Lemma 1 (b) is to be applied. □

It is reasonable to expect that the tree with third-maximal *DRSO*-value is obtained by attaching a new pendent edge to $T_2 \in \mathscr{T}_{n-1}$. This can be done in three different ways, resulting in the trees T_3, T_4, and T_5, cf. the graphs **3,4,5** in Figure 1. Based on their structure, by Eq. (3) we have

$$\begin{aligned}
DRSO(T_3) &= (n-5)\gamma(n-3,1)+\Big[\gamma(n-3,1)+\gamma(n-3,2)+\gamma(2,2)+\gamma(2,1)\Big],\\
DRSO(T_4) &= (n-5)\gamma(n-3,1)+\Big[2\,\gamma(n-3,2)+2\,\gamma(2,1)\Big],\\
DRSO(T_5) &= (n-5)\gamma(n-3,1)+\Big[\gamma(n-3,1)+\gamma(n-3,3)+2\,\gamma(3,1)\Big].
\end{aligned}$$

By direct calculation we obtain, for instance:

$$\gamma(n-3,1)+\gamma(n-3,2)+\gamma(2,2)+\gamma(2,1) = \begin{cases} 6.952 & \text{for} \quad n=5 \\ 9.545 & \text{for} \quad n=7 \\ 12.497 & \text{for} \quad n=9 \\ 22.978 & \text{for} \quad n=16 \end{cases}$$

$$2\,\gamma(n-3,2)+2\,\gamma(2,1) = \begin{cases} 6.952 & \text{for} \quad n=5 \\ 8.246 & \text{for} \quad n=7 \\ 10.160 & \text{for} \quad n=9 \\ 17.127 & \text{for} \quad n=16 \end{cases}$$

$$\gamma(n-3,1)+\gamma(n-3,3)+2\,\gamma(3,1) = \begin{cases} 9.741 & \text{for} \quad n=5 \\ 11.575 & \text{for} \quad n=7 \\ 14.100 & \text{for} \quad n=9 \\ 23.377 & \text{for} \quad n=16 \end{cases}$$

which clearly and undoubtedly implies that for all $n \geq 5$,

$$DRSO(T_5) > DRSO(T_3) > DRSO(T_4)\,.$$

Claim 1. T_5 is the n-vertex tree with third-greatest degree-ratio Sombor index.

4 Trees with Minimal Degree-Ratio Sombor Index

Let $T_6,T_7,T_8,T_9 \in \mathscr{T}_n$ be trees whose 9-vertex representatives **6,7,8,9** are depicted in Figure 2. Note that $T_6 \cong P_n$.

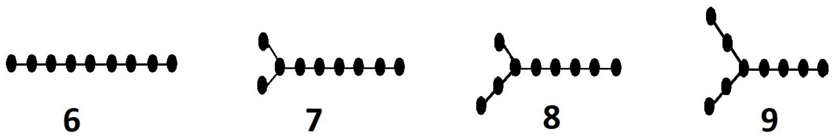

Figure 2. Trees of order 9 with smallest degree-ratio Sombor index.

Theorem 3. *The path $P_n \cong T_6$ is the n-vertex tree with smallest degree-ratio Sombor index.*

Proof. Any tree of order n has $p \geq 2$ pendent and $n-1-p$ non-pendent edges. According to Lemma 1 (a), in order that $DRSO$ be as small as possible, all non-pendent edges must be of (i,i)-type (not necessarily $i=2$). According to Lemma 1 (c), the number of pendent edges should be as small as possible, and $(2,1)$-type pendent edges are preferred. In the case of the path, all these minimality requirements are satisfied: $p=2$, both pendent edges are of $(2,1)$-type, and all other edges are of $(2,2)$-type. Any other tree violates at least one of these properties. Therefore, if $T \in \mathscr{T}_n$ and $T \ncong P_n$, then $DRSO(T) > DRSO(P_n)$. $\square$

It is reasonable to expect that the tree with second-minimal $DRSO$-value possesses a single vertex of degree 3, which we call *branching point.* Such trees possess exactly 3 pendent edges. The set of such n-vertex trees will be denoted by $\mathscr{T}_n^{[3]}$.

$\mathscr{T}_4^{[3]}$ and $\mathscr{T}_5^{[3]}$ have a single element each, T_A and T_B, whereas there are two trees T_C and T_D, belonging to $\mathscr{T}_6^{[3]}$. These are depicted in Figure 3.

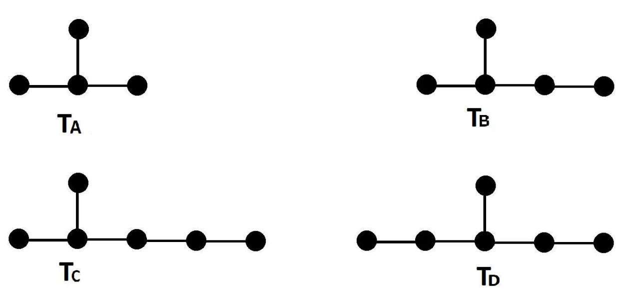

Figure 3. Small trees with a single branching point.

By the following direct calculation we see that $DRSO(T_C) < DRSO(T_D)$:

$$\begin{aligned} DRSO(T_C) &= \gamma(2,1)+2\,\gamma(3,1)+\gamma(3,2)+\gamma(2,2) = 11.156 \\ DRSO(T_D) &= 2\,\gamma(2,1)+\gamma(3,1)+2\,\gamma(3,2) = 12.428\,. \end{aligned}$$

If $n \geq 7$, then the set $\mathscr{T}_n^{[3]}$ can be divided into three parts $\mathscr{T}_n^{[112]}$, $\mathscr{T}_n^{[122]}$, and $\mathscr{T}_n^{[222]}$. The subset $\mathscr{T}_n^{[112]}$ consists of trees in which the neighbors of the branching point have degrees 1, 1, and 2. The meaning of the symbols [122] and [222] is analogous. The graphs **7, 8**, and **9** in Figure 2 are members of the subsets $\mathscr{T}_9^{[112]}$, $\mathscr{T}_9^{[122]}$, and $\mathscr{T}_9^{[222]}$, respectively.

Let $n \geq 7$ and $T_7 \in \mathscr{T}_n^{[112]}$, $T_8 \in \mathscr{T}_n^{[122]}$, $T_9 \in \mathscr{T}_n^{[222]}$. Then

$$\begin{aligned} DRSO(T_7) &= (n-6)\gamma(2,2)+\gamma(2,1)+\gamma(3,2)+\Big[2\,\gamma(3,1)+2\,\gamma(2,2)\Big], \\ DRSO(T_8) &= (n-6)\gamma(2,2)+\gamma(2,1)+\gamma(3,2)+\Big[\gamma(2,1)+\gamma(3,1)+\gamma(3,2)+\gamma(2,2)\Big], \\ DRSO(T_9) &= (n-6)\gamma(2,2)+\gamma(2,1)+\gamma(3,2)+\Big[2\,\gamma(2,1)+2\,\gamma(3,2)\Big]\,. \end{aligned}$$

Bearing in mind that

$$\begin{aligned} 2\,\gamma(3,1)+2\,\gamma(2,2) &= 8.865 \\ \gamma(2,1)+\gamma(3,1)+\gamma(3,2)+\gamma(2,2) &= 8.137 \\ 2\,\gamma(2,1)+2\,\gamma(3,2) &= 7.409, \end{aligned}$$

we conclude that $DRSO(T_9) < DRSO(T_8) < DRSO(T_7)$. Interestingly, $DRSO(T_8) = [DRSO(T_7) + DRSO(T_9)]/2$.

The above findings can be summarized in the following theorem.

Theorem 4.
(a) *If $n = 4$, $n = 5$, and $n = 6$, then the trees T_A, T_B, and T_C (depicted in Figure 3) have the smallest degree-ratio Sombor index in $\mathscr{T}_n^{[3]}$.*
(b) *If $n \geq 7$, then the elements of the subset $\mathscr{T}_n^{[222]}$ (which all have mutually equal DRSO-values) have the smallest degree-ratio Sombor index in $\mathscr{T}_n^{[3]}$.*

Claim 2. The trees specified in Theorem 4 are the trees with the second-smallest degree-ratio Sombor index.

5 Conclusions

What first needs to be noted is that in spite of the significant difference in the algebraic form and geometric interpretation of Sombor and degree-ration Sombor indices, Eqs. (1) and (2), the species having their extremal values appear to be closely similar. Here we have shown this for trees. Recall that the extremality of P_n and S_n w.r.t. Sombor index is a long-time established fact [2, 4].

The case of unicyclic graphs is quite simple. The unicyclic graph with maximal *DRSO* is the star to which a new edge is added, forming a triangle. In view of Remark 1, the cycle (the regular graph of degree 2) is the connected unicyclic graph with minimal *DRSO*. These graphs coincide with the *SO*-extremal species [4].

Finding bicyclic and tricyclic graphs extremal w.r.t. *DRSO* might be much more difficult, and is left as a task for future research. Also, a formal proof of Claims 1 and 2 may remain as an interesting mathematical exercise.

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