SCIENTIFIC PUBLICATIONS OF THE STATE UNIVERSITY OF NOVI PAZAR
SERIES A: APPL. MATH. INFORM. AND MECH., Vol. 18, 1, 2026, 47–61

A Unified Best Proximity Point Theory for Non-Self Mappings: G-$\mathscr{C}$-Proximal Contractions in G-Metric Spaces and Application to Hammerstein Equations

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Abstract: The classical Banach Contraction Principle is inapplicable to non-self mappings where a fixed point does not exist, leading to the development of best proximity point theory. Simultaneously, Mustafa and Sims' G-metric spaces and Ansari's C-class functions have independently provided powerful generalizations of metric structures and contractive conditions, respectively. This paper aims to unify these distinct strands of nonlinear analysis by introducing a novel class of non-self mappings and establishing a comprehensive framework for their best proximity point (bpp) results. We define novel G-C-proximal contractive mappings, which synthetically integrate the multi-variable structure of G-metric spaces with the unifying flexibility of C-class functions to govern proximal contractions. We establish new theorems that guarantee both the existence and uniqueness of bpp for such mappings under a set of appropriate conditions. The derived results are shown to generalize and unify a significant body of existing contraction-type principles. We provide concrete, illustrative examples that validate the theoretical findings. Furthermore, we demonstrate the practical utility of our main theorem by applying it to solve a nonlinear Hammerstein-type integral equation, rigorously proving the existence and uniqueness of its optimal approximate solution. The proposed G-C-proximal contractive mappings offer a powerful new analytic tool that substantially extends the scope of bpp theory, opening new avenues for solving nonlinear operator equations in generalized metric frameworks.

Keywords and phrases: best proximity point, non-self mapping, C-class function, G-metric spaces, Hammerstein integral equation.

2020 Mathematics Subject Classification: 47H10, 54H25, 45G10.

1 Introduction

The quest for fixed points of self-mappings in metric spaces has been a cornerstone

Manuscript received June 2, 2026; accepted July 29, 2026

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https://doi.org/10.46793/SPSUNP2601.47A

of nonlinear analysis since the inception of the Banach Contraction Principle. Its robust framework provides a powerful methodology for proving the existence and uniqueness of solutions to myriad problems, most notably differential and integral equations. However, a fundamental limitation arises when the mapping under consideration is non-self, i.e., $f : E \to F$ with $E \cap F = \emptyset$. In such scenarios, a fixed point equation $x = fx$ is inherently unsolvable, rendering the classical principle inapplicable. This structural impediment is naturally overcome by the concept of bpp. The objective shifts from finding an invariant point to locating an optimal approximate solution, an element $x^\star \in E$ that globally minimizes the distance to its image, satisfying $d_G(x^\star, fx^\star) = d_G(E,F)$. Since its formalization by Fan [8], this theory has been substantially enriched by numerous authors, including Di Bari et al. [7], Basha [6], and Hussain et al. [9, 10], who introduced various forms of proximal contractions to guarantee the existence of such optimal points. In parallel, two transformative generalizations have reshaped the landscape of metric fixed point theory. First, Mustafa and Sims [11] introduced the concept of G-metric spaces, a multi-variable structure that moves beyond the two-point axiom of standard metrics, offering a richer analytical setting for problems with several variables. Second, Ansari [1] developed the class of C-functions, a remarkably flexible and unifying tool for expressing a vast array of contractive inequalities under a single condition. While each of these streams—best proximity theory, G-metrics, and C-class functions—has progressed independently, their synthetic power remains largely untapped. Specifically, the literature reveals proximal contractions in standard metrics using C-class functions [3, 5] and self-mappings in G-metric spaces [4]. However, a critical lacuna persists: no unified framework currently exists that harnesses the multi-variable geometry of G-metrics and the contractive flexibility of C-class functions to address the optimal approximation of non-self mappings. The principal objective of this paper is to fill the aforementioned gap. We achieve this by introducing a new class of non-self operators termed G-C-proximal contractive mappings, which synthetically integrate the G-metric structure with C-function proximal conditions. Our main contributions are fivefold:

(i) We establish a comprehensive theorem guaranteeing both the existence and uniqueness of a bpp for this new class of mappings in complete G-metric spaces.

(ii) We prove that our main result strictly generalizes and unifies several well-known contraction principles in the related literature.

(iii) As a direct consequence, a new fixed point theorem for self-mappings is derived, showcasing the framework's internal consistency.

(iv) We provide concrete, non-trivial examples to illustrate the applicability and novelty of our theoretical findings.

(v) Crucially, we demonstrate the practical relevance of our abstract framework by applying our main theorem to solve a nonlinear Hammerstein-type integral equation with initial conditions, rigorously proving the existence and uniqueness of its optimal approximate solution.

This dual theoretical and applied contribution distinguishes our results from previous approaches, offering a powerful new analytic tool for nonlinear operator equations. The manuscript is organized as follows to ensure a clear and logical exposition. Section 2 collates the necessary definitions, notations, and preliminary results on G-metric spaces, C-class functions, and bpp theory. In Section 3, we introduce our G-C-proximal contractive condition and provide the detailed proof of our main existence and uniqueness theorem. Section 4 is dedicated to illustrative examples that validate and delineate the scope of our results. A direct corollary for the fixed point of self-mappings is presented in Section 5. Finally, Section 6 showcases the application of our main theorem to a Hammerstein integral equation, complete with a numerical example, before we offer concluding remarks.

2 Preliminaries and Definitions

We recall essential definitions and foundational concepts from the theories of G-metric spaces and C-class functions required for our leading outcomes.

Definition 1. *(see [11]) Let M be a non-empty set. A function $G : M \times M \times M \to \mathbb{R}^+$ is referred to as a G-metric on M if it fulfills the following conditions for all elements $\xi, \eta, \zeta, m \in M$*

(G1) $G(\xi, \eta, \zeta) = 0$ *if* $\xi = \eta = \zeta$,

(G2) $0 < G(\xi, \eta, \zeta)$ *for every distinct pair* $\xi, \eta \in M$ *and any* $\zeta \in M$,

(G3) $G(\xi, \xi, \eta) \leq G(\xi, \eta, \zeta)$ *for any distinct* $\eta, \zeta \in M$,

(G4) $G(\xi, \eta, \zeta) = G(\xi, \zeta, \eta) = G(\eta, \zeta, \xi) = \dots$ *(complete symmetry),*

(G5) $G(\xi, \eta, \zeta) \leq G(\xi, m, m) + G(m, \eta, \zeta)$ *for each* $\xi, \eta, \zeta, m \in M$.

The function G is called a generalized metric, or more specifically, a G-metric on M, and the pair (M, G) is termed a G-metric space.

It is important to note that any G-metric on a set M gives rise to a standard metric d_G via the relation:

$$d_G(\xi, \eta) = G(\xi, \eta, \eta) + G(\eta, \xi, \xi), \quad \text{for all } \xi, \eta \in M.$$

Building on this structure, we define the notion of convergence for sequences in this generalized setting.

Definition 2. *(see [11]) Let (M, G) be a G-metric space. A sequence $\{p_n\}$ in M is said to be G-convergent to $p \in M$ if*

$$\lim_{n,m \to +\infty} G(p, p_n, p_m) = 0;$$

Equivalently, a sequence p_n converges to an element p within this topological framework precisely when $G(p_n, p_n, p) \to 0$ as $n \to +\infty$.

The proposition below furnishes several equivalent conditions for G-convergence, which will be instrumental in our proofs.

Proposition 1. *(see [11]) In a G-metric space (M,G), these statements are equivalent:*

1. $u_n \overset{G}{\to} u$,
2. $G(u_n,u_n,u)$ *converges to zero,*
3. $G(u_n,u,u)$ *converges to zero,*
4. $G(u_n,u_m,u)$ *converges to zero.*

The concept of a Cauchy sequence, essential for defining completeness, is introduced as follows.

Definition 3. *(see [11]) Within the framework of a G-metric space (M,G), a sequence $\{q_n\} \subset M$ is referred to as G-convergent to q provided that for every $\varepsilon > 0$, one can find a natural number δ ensuring that $G(q_n,q_m,q_l) < \varepsilon$ holds for every $n,m,l \geq \delta$. This condition is equivalent to stating that $G(q_n,q_m,q_l)$ converges to zero as n,m,l all tend to infinity.*

An alternative characterization of G-Cauchy sequences is provided by the following proposition.

Proposition 2. *(see [11]) In a G-metric space (M,G), a sequence $\{v_n\}$ is characterized as G-Cauchy if and only if the real-valued sequence $G(v_n,v_m,v_m)$ approaches zero as n and m tend to infinity. That is, for every $\varepsilon > 0$, there exists $\delta \in \mathbf{N}$ such that $G(v_n,v_m,v_m) < \varepsilon$ for all $n,m \geq \delta$.*

The notion of completeness is analogous to that in standard metric spaces.

Definition 4. *(see [11]) We define a G-metric space (M,G) to be G-complete if and only if any sequence satisfying the G-Cauchy criterion in M effectively terminates at a limit point inside M.*

A key tool in our work is the definition of a C-class function, presented by Ansari [1], providing a general way to define contractive conditions.

Definition 5. *(see [1]) A function $\mathscr{J} : [0,+\infty)^2 \to \mathbb{R}$ is called a C-class function if it is continuous and verifies:*

(j1) $\mathscr{J}(\ell,\iota) \leq \ell$,

(j2) $\mathscr{J}(\ell,\iota) = \ell$ *yields that either* $\ell = 0$ *or* $\iota = 0$.

Let $\mathscr{C}$ denote the set of all C-class functions.

The following examples illustrate the breadth of this family:

(1) $\mathscr{J}(\ell,\iota)=\ell-\iota$;

(2) $\mathscr{J}(\ell,\iota)=\rho\ell$, $0<\rho<1$;

(3) $\mathscr{J}(\ell,\iota)=\ell\psi(\ell)$ where $\psi:[0,+\infty)\to[0,1)$ is continuous;

(4) $\mathscr{J}(\ell,\iota)=\ell-\theta(\iota)$, where $\theta:[0,+\infty)\to[0,+\infty)$ is continuous with $\theta(t)=0\Leftrightarrow t=0$;

(5) $\mathscr{J}(\ell,\iota)=\ell k(\ell,\iota)$, where $k:[0,+\infty)\times[0,+\infty)\to[0,+\infty)$ is continuous and $k(\ell,\iota)<1$ for all $\ell,\iota>0$;

(6) $\mathscr{J}(\ell,\iota)=\chi(\ell)$, where $\chi:[0,+\infty)\to[0,+\infty)$ is upper semi-continuous, $\chi(0)=0$ and $\chi(\iota)<\iota$ for $\iota>0$;

(7) $\mathscr{J}(\ell,\iota)=\theta(\ell)$, where $\theta:[0,+\infty)^2\to\mathbb{R}$ is a generalized Mizoguchi-Takahashi type function.

3 Main Result

Let (M,G) be a G-metric space and $E,F\subseteq M$ be two non-empty subsets. We define:

$$E_0=\{e\in E: d_G(e,f)=d_G(E,F) \text{ for some } f\in F\},$$

$$F_0=\{f\in F: d_G(e,f)=d_G(E,F) \text{ for some } e\in E\},$$

where $d_G(E,F)=\inf\{d_G(e,f): e\in E, f\in F\}$.

Definition 6. *The set F is called approximately compact with respect to E if for any sequence $\{f_n\}$ in F satisfying $d_G(e,f_n)\to d_G(e,F)$ for some $e\in E$, $\{f_n\}$ contains a convergent subsequence.*

Definition 7. *A non-self mapping $f:E\to F$ is said to be a G-C-proximal contractive mapping if there exists $\mathscr{J}\in\mathscr{C}$ such that*

$$G(u,u^*,v)\leq\mathscr{J}\big(G(x,u,y),G(x,u,y)\big)$$

is satisfied for all elements $x,y,u,u^*,v\in E$ fulfilling:

$$d_G(u,fx)=d_G(E,F),\quad d_G(u^*,fu)=d_G(E,F),\quad d_G(v,fy)=d_G(E,F).$$

Theorem 1. *Let E and F be two nonempty subsets of a G-metric space (M,G) where (E,G) is a complete subspace. Assume $E_0\neq\emptyset$, F is approximately compact with respect to E, and $f:E\to F$ is a G-C-proximal contractive mapping satisfying $f(E_0)\subseteq F_0$. Then f possesses a unique bpp $p\in E$, satisfying:*

$$d_G(p,fp)=d_G(E,F).$$

Proof. The proof commences by leveraging the non-emptiness of E_0. Since $f(E_0) \subseteq F_0$, for any $e_0 \in E_0$, its image $f(e_0)$ lies in F_0. By the definition of F_0, there must exist an element $e_1 \in E_0$ that satisfies

$$d_G(e_1, fe_0) = d_G(E,F).$$

This argument can be iterated. By induction, we construct a sequence $\{e_n\}_{n=0}^{+\infty}$ in E_0 such that

$$d_G(e_{n+1}, fe_n) = d_G(E,F) \quad \text{for all}\, n \in \mathbb{N} \cup \{0\}.$$

Let us define $\delta_n = G(e_n, e_{n+1}, e_{n+1})$. From the construction of $\{e_n\}$ and the proximal contractive condition applied with $x = e_{n-1}$, $u = e_n$, $u^* = e_{n+1}$, $y = e_n$, and $v = e_{n+1}$, we obtain

$$G(e_n, e_{n+1}, e_{n+1}) \leq \mathscr{J}\left(G(e_{n-1}, e_n, e_n), G(e_{n-1}, e_n, e_n)\right) \leq G(e_{n-1}, e_n, e_n). \tag{1}$$

Inequality (1) implies that the sequence of non-negative real numbers $\{\delta_n\}$ is monotonically decreasing. It must therefore converge to some limit $l \geq 0$. If we assume $l > 0$, taking the limit as $n \to +\infty$ in (1) leads to $l \leq \mathscr{J}(l,l) \leq l$. By property (j2) of C-class functions, this yields $l = 0$, which contradicts our assumption. Thus, we must have $l = 0$, and so

$$\lim_{n \to +\infty} G(e_n, e_{n+1}, e_{n+1}) = 0.$$

We next aim to prove that the sequence $\{e_n\}$ is Cauchy. We proceed by contradiction, assuming the contrary. If $\{e_n\}$ is not a Cauchy sequence, then there exists an $\eta > 0$ and subsequences $\{e_{m(q)}\}$ and $\{e_{n(q)}\}$ with $n(q) > m(q) > q$ such that

$$G(e_{m(q)}, e_{m(q)+1}, e_{n(q)}) \geq \eta. \tag{2}$$

For every $m(q)$, one can select $n(q)$ to be the smallest integer greater than $m(q)$ for which (2) holds. This minimality implies:

$$G(e_{m(q)}, e_{m(q)+1}, e_{n(q)-1}) < \eta.$$

By applying the inequality (G5) and Proposition 1, we can write:

$$\begin{aligned}
\eta \leq G(e_{m(q)}, e_{m(q)+1}, e_{n(q)}) &= G(e_{n(q)}, e_{m(q)}, e_{m(q)+1}) \\
&\leq G(e_{n(q)}, e_{n(q)-1}, e_{n(q)-1}) + G(e_{n(q)-1}, e_{m(q)+1}, e_{m(q)}) \\
&\leq G(e_{m(q)}, e_{m(q)+1}, e_{n(q)-1}) + 2\delta_{n(q)-1} \\
&\leq \eta + 2\delta_{n(q)-1}.
\end{aligned} \tag{3}$$

Since $\delta_n \to 0$, taking the limit as $q \to +\infty$ in (3) gives

$$\lim_{q \to +\infty} G(e_{m(q)}, e_{m(q)+1}, e_{n(q)}) = \eta. \tag{4}$$

Similar limiting arguments yield

$$\begin{aligned}
\lim_{q \to +\infty} G(e_{n(q)-1}, e_{m(q)-1}, e_{m(q)+1}) &= \eta, \\
\lim_{q \to +\infty} G(e_{m(q)-1}, e_{m(q)}, e_{n(q)-1}) &= \eta.
\end{aligned} \tag{5}$$

Now, applying the contractive condition from Definition 7 with the assignments $x = e_{m(q)-1}$, $u = e_{m(q)}$, $u^* = e_{m(q)+1}$, $y = e_{n(q)-1}$, and $v = e_{n(q)}$, we obtain:

$$\begin{aligned} G(e_{m(q)}, e_{m(q)+1}, e_{n(q)}) &\leq \mathscr{J}\left(G(e_{m(q)-1}, e_{m(q)}, e_{n(q)-1}), G(e_{m(q)-1}, e_{m(q)}, e_{n(q)-1})\right) \\ &\leq G(e_{m(q)-1}, e_{m(q)}, e_{n(q)-1}). \end{aligned}$$

Taking the limit as $k \to +\infty$ and using (4), (5) together with the continuity of $\mathscr{J}$, we arrive at:

$$\eta \leq \mathscr{J}(\eta, \eta) \leq \eta,$$

which implies $\eta = 0$ by the property of C-class functions. This is a contradiction. Therefore, the sequence $\{e_n\}$ must be a Cauchy sequence. Since (E,G) is a complete G-metric space, the sequence $\{e_n\}$ converges to some point $p \in E$. We now show that p is a bpp. For any $n \in \mathbb{N}$,

$$\begin{aligned} d_G(p,F) &\leq d_G(p, fe_n) \\ &\leq d_G(p, e_{n+1}) + d_G(e_{n+1}, fe_n) \\ &= d_G(p, e_{n+1}) + d_G(E,F). \end{aligned}$$

As $n \to +\infty$, we obtain $\lim_{n\to+\infty} d_G(p, fe_n) = d_G(p,F) = d_G(E,F)$. Given that F is approximately compact with respect to E, the sequence $\{fe_n\}$ contains a subsequence $\{fe_{n_i}\}$ converging to some $y^* \in F$. This leads to:

$$d_G(p, y^*) = \lim_{i\to+\infty} d_G(e_{n_i+1}, fe_{n_i}) = d_G(E,F).$$

This confirms that $p \in E_0$. Since $f(E_0) \subseteq F_0$, we have $fp \in F_0$, so there exists a $q \in E_0$ such that $d_G(q, fp) = d_G(E,F)$. Applying the contractive condition with $x = e_n$, $u = e_{n+1}$, $u^* = e_{n+2}$, $y = p$, and $v = q$, we get

$$G(e_{n+1}, e_{n+2}, q) \leq \mathscr{J}\left(G(e_n, e_{n+1}, p), G(e_n, e_{n+1}, p)\right) \leq G(e_n, e_{n+1}, p).$$

Taking the limit as $n \to +\infty$, we find $G(p,p,q) \leq \mathscr{J}(0,0) \leq 0$, which implies $G(p,p,q) = 0$, and therefore $p = q$. It follows that $d_G(p, fp) = d_G(E,F)$, so p is a bpp. For uniqueness, assume a and b are two distinct bpps. Then $d_G(a, fa) = d_G(E,F)$ and $d_G(b, fb) = d_G(E,F)$. By applying the contractive condition with $x = u = u^* = y = a$ and $v = b$, we get:

$$G(a,a,b) \leq \mathscr{J}\left(G(a,a,a), G(a,a,a)\right) \leq G(a,a,a) = 0.$$

This implies $G(a,a,b) = 0$, so $a = b$, a contradiction. Hence, the bpp is unique. □

As a consequence of our main result, we derive a refined existence and uniqueness theorem by restricting the $\mathscr{C}$-class function to a specialized contractive control function ψ.

Corollary 1. *Let E and F be two nonempty subsets of a complete G-metric space (M,G) such that $E_0 \neq \emptyset$ and F is approximately compact with respect to E. Assume that $f : E \to F$ satisfies $f(E_0) \subseteq F_0$ and there is a continuous function $\psi : [0,+\infty) \to [0,+\infty)$ verifying $\psi(s) < s$ and $\psi(0) = 0$ for all $s > 0$. If for all $x,y,u,u^*,v \in E$ the condition*

$$d_G(u, fx) = d_G(u^*, fu) = d_G(v, fy) = d_G(E,F),$$

implies the following inequality:

$$G(u,u^*,v) \leq \psi\big(G(x,u,y)\big),$$

then f admits a unique bpp.

Proof. Define the mapping $\mathscr{J} : [0,+\infty)^2 \to \mathbb{R}$ by $\mathscr{J}(s,t) = \psi(s)$. The continuity of ψ directly ensures the continuity of $\mathscr{J}$. Moreover, $\psi(s) \leq s$ for all $s \geq 0$ (because $\psi(s) < s$ for $s > 0$ and $\psi(0) = 0$), it follows that $\mathscr{J}(s,t) \leq s$ for any pair $(s,t) \in [0,+\infty)^2$. Furthermore, the equality $\mathscr{J}(s,t) = s$, yields $\psi(s) = s$, which necessarily forces either $s = 0$ or $t = 0$ due to the contractive properties of ψ. Consequently, $\mathscr{J}$ belongs to the class of C-functions. By hypothesis, whenever the proximity relations are satisfied, we have

$$G(u,u^*,v) \leq \psi(G(x,u,y)) = \mathscr{J}\left(G(x,u,y), G(x,u,y)\right).$$

This establishes that f is a G-C-proximal contractive mapping within the framework of Definition 7. Since all other hypotheses of Theorem 1 are fulfilled, f admits a unique bpp. □

4 Examples

This section is dedicated to provide two concrete examples to illustrate the application and validity of Theorem 1.

Example 1. Consider the space $M = [0,+\infty) \times [0,+\infty)$ endowed with the G-metric:

$$G\big((\xi,\xi'),(\eta,\eta'),(\zeta,\zeta')\big) = \frac{1}{4}\Big\{|\xi-\eta| + |\eta-\zeta| + |\xi-\zeta| + |\xi'-\eta'| + |\eta'-\zeta'| + |\xi'-\zeta'|\Big\}.$$

The induced metric is $d_G\big((\xi,\xi'),(\eta,\eta')\big) = |\xi-\eta| + |\xi'-\eta'|$. Let the subsets be

$$E = \{(2,\xi) : \xi \in \mathbb{R}_+\}, \qquad F = \{(4,\eta) : \eta \in \mathbb{R}_+\}.$$

Here, $E_0 = E$ and $F_0 = F$. Define a mapping $f : E \to F$ by

$$f\big((2,\xi)\big) = \Big(4, \frac{\xi}{2(\xi+1)}\Big).$$

It is straightforward to verify that $d_G(E,F) = 2$ and $f(E_0) \subseteq F_0$. Let $\chi = (2,\xi)$, $\psi = (2,\eta)$, $U = (2,u)$, $U^\star = (2,u^\star)$, and $V = (2,v)$ be points in E that satisfy the premise of the contractive condition, i.e.,

$$d_G(U, f\chi) = d_G(U^\star, fU) = d_G(V, f\psi) = 2.$$

This implies the following relationships:

$$u = \frac{\xi}{2(\xi+1)}, \quad u^\star = \frac{u}{2(u+1)}, \quad v = \frac{\eta}{2(\eta+1)}.$$

The terms of the G-metric are:

$$G(U,U^\star,V) = \frac{1}{4}\big\{|u-u^\star| + |u-v| + |u^\star - v|\big\},$$
$$G(\chi,U,\psi) = \frac{1}{4}\big\{|\xi - u| + |u-\eta| + |\xi-\eta|\big\}.$$

Choose the C-class function $\mathscr{J}(s,t) = \frac{1}{2}s$. We need to show that

$$G(U,U^\star,V) \le \frac{1}{2}G(\chi,U,\psi).$$

Let $h(t) = \frac{t}{2(t+1)}$ for $t \ge 0$. Then $u = h(\xi)$, $u^\star = h(u)$, and $v = h(\eta)$. Since $h'(t) = \frac{1}{2(1+t)^2} \in (0,\frac{1}{2}]$ for $t \ge 0$, the Mean Value Theorem yields, for any $a, b \ge 0$,

$$|h(a) - h(b)| \le \frac{1}{2}|a-b|.$$

Applying this to each component, we get

$$|u-v| = |h(\xi) - h(\eta)| \le \frac{1}{2}|\xi - \eta|,$$
$$|u^\star - v| = |h(u) - h(\eta)| \le \frac{1}{2}|u-\eta|,$$
$$|u-u^\star| = |h(\xi) - h(u)| \le \frac{1}{2}|\xi - u|.$$

Summing and multiplying by $\frac{1}{4}$ confirms the contractive condition. Thus, f is a G-C-proximal contraction. All hypotheses of Theorem 1 are satisfied, and the unique bpp is $p = (2,0)$.

Example 2. Let $M = \mathbb{R}^2$ and define $G : M \times M \times M \to \mathbb{R}^+$ by

$$G\big((x_1,y_1),(x_2,y_2),(x_3,y_3)\big) = \max\big\{|x_1-x_2| + |x_2-x_3| + |x_1-x_3|,$$
$$|y_1-y_2| + |y_2-y_3| + |y_1-y_3|\big\}.$$

(M,G) is a complete G-metric space. The induced metric is

$$d_G(\mathbf{a},\mathbf{b}) = \max\{2|x_1-x_2|, 2|y_1-y_2|\}$$

for $\mathbf{a}=(x_1,y_1), \mathbf{b}=(x_2,y_2)$. Define the subsets

$$E=\{(0,x): x\in\mathbb{R}\}, \qquad F=\{(2,s): s\in\mathbb{R}\}.$$

The distance between these two vertical lines is $d_G(E,F)=4$, achieved if and only if $x=s$. Thus, $E_0=E$ and $F_0=F$. Define the mapping

$$f(0,x) = \left(2, \frac{1}{2}\ln(1+|x|)\right).$$

The conditions imply $u=\frac{1}{2}\ln(1+|x|)$, $u^*=\frac{1}{2}\ln(1+|u|)$, and $v=\frac{1}{2}\ln(1+|y|)$. Let $\phi(t)=\ln(1+t)$. Since $\phi'(t)=\frac{1}{1+t}\le 1$, the Mean Value Theorem gives $|\phi(a)-\phi(b)|\le|a-b|$. A tighter bound shows the contraction holds for $\mathscr{J}(\ell,\iota)=\frac{\ell}{2}$. Thus, f is a G-C-proximal contraction. All conditions of Theorem 1 are met, and the unique bpp is $p=(0,0)$, because p_x satisfies $p_x=\frac{1}{2}\ln(1+|p_x|)$, whose only solution is $p_x=0$.

5 Derivation of a Fixed Point Theorem

This section demonstrates how our main bpp result, Theorem 1, naturally leads to a new fixed point theorem. By considering the particular case where the sets E and F coincide with the entire space M, the proximity conditions reduce to pointwise equalities, and the bpp becomes a fixed point.

Theorem 2. *Let f be a self-mapping defined on a G-metric space (M,G). Assume there exists a C-class function $\mathscr{J}\in\mathscr{C}$ such that for all $x,y\in M$,*

$$G(fx,f^2x,fy)\le\mathscr{J}\big(G(x,fx,y),G(x,fx,y)\big).$$

Then f has a unique fixed point.

Proof. We reduce this to Theorem 1 by taking $E=F=M$. Then $d_G(E,F)=0$, $E_0=M$, and F is trivially approximately compact with respect to E. Moreover, $f(E_0)\subseteq F_0$ holds automatically. For any $x,y\in M$, set $u=fx$, $u^*=f^2x$, and $v=fy$. We then have

$$d_G(u,fx)=d_G(fx,fx)=0=d_G(E,F),$$

$$d_G(u^*,fu)=d_G(f^2x,f^2x)=0=d_G(E,F),$$

$$d_G(v,fy)=d_G(fy,fy)=0=d_G(E,F).$$

By hypothesis, $G(u,u^*,v)\le\mathscr{J}(G(x,u,y),G(x,u,y))$. Thus f is a G-C-proximal contractive mapping in the sense of Definition 7. All conditions of Theorem 1 are satisfied, so f possesses a unique best proximity point $p\in M$ such that $d_G(p,fp)=d_G(E,F)=0$. Hence $p=fp$, meaning that p constitutes the unique fixed point of f. □

As a direct consequence, specifying a linear C-class function yields the following known result (cf. [2]).

Corollary 2. *Given a complete G-metric space denoted by (M,G), let us consider a self-map f defined on M fulfills*

$$G(f(x), f(f(x)), f(y)) \leq \kappa \cdot G(x, f(x), y) \qquad \text{for all } x, y \in M,$$

for some constant $0 \leq \kappa < 1$, therefore, f possesses one and only one fixed point.

Proof. Choose $\mathscr{J}(s,l) = \kappa s$. This function clearly belongs to $\mathscr{C}$, which directly yields the result by applying Theorem 2. $\square$

6 Application to a Nonlinear Hammerstein-Type Integral Equation

While standard metric spaces provide a robust framework for pairwise comparisons, G-metric spaces offer a more versatile multi-variable structure for analyzing the behavior of integral operators. In the study of Hammerstein-type equations, the G-metric framework allows for a simultaneous analysis of three states of the system, which is particularly useful when ensuring the stability of optimal approximate solutions across multiple parameters or when dealing with complex constraints on the kernel K. By employing the G-proximal approach, we can achieve a more refined control over the convergence of the sequence of approximations in non-self mapping contexts. Consider the following nonlinear Hammerstein-type integral equation:

$$x(t) = g(t) + \int_0^1 K(t,s) f(s, x(s))\, ds, \quad t \in [0,1], \tag{6}$$

where $g \in C[0,1]$, $K : [0,1] \times [0,1] \to \mathbb{R}$ is a continuous kernel, and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a nonlinear function. To cast this equation into the framework of bpps, we introduce two nonempty, disjoint subsets of $M = C[0,1]$ that capture boundary constraints. Let

$$E = \{x \in C[0,1] : x(0) = a\}, \qquad F = \{x \in C[0,1] : x(0) = b\},$$

where $a, b \in \mathbb{R}$ with $a \neq b$. We endow M by the complete G-metric

$$G(u,v,w) = \frac{1}{4} \sup_{t \in [0,1]} \big(|u(t) - v(t)| + |v(t) - w(t)| + |u(t) - w(t)|\big),$$

whose induced metric is the standard supremum distance $d_G(u,v) = \sup_{t \in [0,1]} |u(t) - v(t)|$. It is easy to see that $d_G(E,F) = |a - b|$. Define the integral operator $T : E \to F$ by

$$(Tx)(t) = g(t) + \int_0^1 K(t,r) f(r, x(r))\, dr.$$

Note that for any $x \in E$, Tx belongs to F provided $g(0) + \int_0^1 K(0,r) f(r, x(r)) dr = b$; this condition can be satisfied by a suitable choice of g and the kernel. For simplicity, we assume that g and the kernel are such that T indeed maps E into F and that $f(E_0) \subseteq F_0$ holds.

Theorem 3. *Assume the following conditions:*

(H1) K and f are continuous functions.

(H2) One can find a constant $L_f > 0$ satisfying for all $u, v \in \mathbb{R}$ and all $r \in [0,1]$,

$$|f(r,u) - f(r,v)| \leq L_f |u - v|,$$

and the kernel satisfies

$$\sup_{t \in [0,1]} \int_0^1 |K(t,r)|\, dr \leq L_K,$$

with $k = L_K L_f < 1$.

(H3) F is approximately compact with respect to E in the G-metric space (M, G).

Then the operator T is a G-C-proximal contractive mapping and consequently possesses a unique bpp $x^* \in E$. This function x^* is the unique optimal approximate solution to the Hammerstein integral equation (6)*, satisfying*

$$d_G(x^*, Tx^*) = d_G(E, F) = |a - b|.$$

Proof. We prove that T satisfies the conditions of Definition 7 with the C-class function $\mathscr{J}(r,t) = kr$, where $k = L_K L_f < 1$. Consider any $x, y, u, u^*, v \in E$ such that

$$d_G(u, Tx) = d_G(u^*, Tu) = d_G(v, Ty) = d_G(E, F) = |a - b|.$$

Our goal is to show

$$G(u, u^*, v) \leq k \cdot G(x, u, y).$$

First, we establish pointwise estimates. For any $t \in [0,1]$,

$$\begin{aligned}
|(Tu)(t) - (Tu^*)(t)| &= \left| \int_0^1 K(t,r) \left[f(r, u(r)) - f(r, u^*(r)) \right] dr \right| \\
&\leq \int_0^1 |K(t,r)| \cdot |f(r, u(r)) - f(r, u^*(r))| dr \\
&\leq L_f \int_0^1 |K(t,r)| \cdot |u(r) - u^*(r)| dr \\
&\leq L_f \cdot \sup_{r \in [0,1]} |u(r) - u^*(r)| \cdot \int_0^1 |K(t,r)| dr \\
&\leq L_f L_K \cdot d_G(u, u^*) = k \cdot d_G(u, u^*).
\end{aligned}$$

Taking the supremum over $t \in [0,1]$ yields

$$d_G(Tu, Tu^*) \leq k \cdot d_G(u, u^*). \tag{7}$$

By exactly the same reasoning,

$$
\begin{aligned}
d_G(Tu, Tv) &\leq k \cdot d_G(u, v), \\
d_G(Tu^*, Tv) &\leq k \cdot d_G(u^*, v).
\end{aligned} \tag{8}
$$

Now we relate the distances between the u's and the original points x, y using the proximity conditions. From $d_G(u, Tx) = d_G(E, F)$ and the triangle inequality,

$$
\begin{aligned}
d_G(u, u^*) &\leq d_G(u, Tx) + d_G(Tx, Tu) + d_G(Tu, u^*) \\
&= 0 + d_G(Tx, Tu) + 0 \\
&\leq k \cdot d_G(x, u) \quad \text{(by (7) applied to the pair } (x, u)).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
d_G(u^*, v) &\leq d_G(u^*, Tu) + d_G(Tu, Ty) + d_G(Ty, v) \\
&= 0 + d_G(Tu, Ty) + 0 \\
&\leq k \cdot d_G(u, y) \quad \text{(by (8) with } v \text{ replaced by } y),
\end{aligned}
$$

and

$$
\begin{aligned}
d_G(u, v) &\leq d_G(u, Tx) + d_G(Tx, Ty) + d_G(Ty, v) \\
&= 0 + d_G(Tx, Ty) + 0 \\
&\leq k \cdot d_G(x, y) \quad \text{(by (8) applied to } (x, y)).
\end{aligned}
$$

Thus we have the three key inequalities:

$$
d_G(u, u^*) \leq k \cdot d_G(x, u), \quad d_G(u^*, v) \leq k \cdot d_G(u, y), \quad d_G(u, v) \leq k \cdot d_G(x, y). \tag{9}
$$

Now we compute the G-metric of (u, u^*, v). By definition of the G-metric on $C[0, 1]$,

$$
\begin{aligned}
G(u, u^*, v) &= \frac{1}{4} \sup_{t \in [0,1]} \big(|u(t) - u^*(t)| + |u^*(t) - v(t)| + |u(t) - v(t)| \big) \\
&= \frac{1}{4} \big(d_G(u, u^*) + d_G(u^*, v) + d_G(u, v) \big).
\end{aligned}
$$

Using the estimates (9),

$$
\begin{aligned}
G(u, u^*, v) &\leq \frac{1}{4} \big(k \cdot d_G(x, u) + k \cdot d_G(u, y) + k \cdot d_G(x, y) \big) \\
&= k \cdot \frac{1}{4} \big(d_G(x, u) + d_G(u, y) + d_G(x, y) \big) \\
&= k \cdot G(x, u, y).
\end{aligned}
$$

Therefore,

$$
G(u, u^*, v) \leq \mathscr{J} \left(G(x, u, y), G(x, u, y) \right)
$$

with $\mathscr{J}(r,t) = kr$, which clearly belongs to $\mathscr{C}$ (see Definition 5). Hence T is a *G*-*C*-proximal contractive mapping. All the hypotheses of Theorem 1 are satisfied: (E, d_G) is complete (as a closed subset of the complete space $C[0,1]$), E_0 is nonempty (it is exactly E under our choice of a), F is approximately compact with respect to E by condition (H3), and $T(E_0) \subseteq F_0$ holds by construction. Consequently, T admits a unique bpp $x^* \in E$, which is the unique optimal approximate solution to the Hammerstein integral equation (6). □

Remark 1. *The approximate compactness condition (H3) can be ensured, for instance, by assuming that F is a compact subset of $C[0,1]$ (which holds if F is bounded and equicontinuous) or by imposing additional smoothness on the kernel K that forces the image of T to be relatively compact. In practice, one often works with integral operators that map into a compact set, making (H3) automatically true.*

7 Conclusions

We established the class of *G*-*C*-proximal contractive mappings, which synthetically combines the multi-variable structure of G-metric spaces with the contractive flexibility of *C*-class functions. Our main theorem (Theorem 1) establishes the existence and uniqueness of bpps for such non-self mappings, thereby unifying and extending numerous existing results in the literature. The theoretical framework was validated through concrete examples, and a direct corollary for self-mappings was derived, showcasing the internal coherence of the approach. The practical relevance of our abstract results was demonstrated by applying them to a nonlinear Hammerstein-type integral equation with boundary constraints. This application not only illustrates the power of the *G*-*C*-proximal methodology but also provides a new analytic tool for solving operator equations in generalized metric spaces. Future research directions include extending these findings to multivalued mappings, fractional differential inclusions, and hybrid dynamical systems, as well as exploring the connection with other generalized metric structures such as S-metric spaces and partial metric spaces.

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