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Some New Lower Bounds on Laplacian Resolvent Energy of Graphs

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Abstract: Let G be a simple connected graph of order $n \geq 3$ with m edges and the vertex degree sequence $d_1 \geq d_2 \geq \cdots \geq d_n$. The Laplacian resolvent energy of G is defined as $RL(G) = \sum_{i=1}^{n} \frac{1}{n+1-\mu_i}$, where $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} > \mu_n = 0$, are Laplacian eigenvalues of G. In this paper, we find some new lower bounds for $RL(G)$ involving the vertex degrees d_1, d_2, d_{n-1}, d_n and various degree-based graph invariants.

Keywords and phrases: Laplacian eigenvalues, Laplacian resolvent energy, vertex degrees.
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1 Introduction

Let $G = (V,E)$, $V = \{v_1, v_2, \ldots, v_n\}$, be a simple connected graph of order $n \geq 3$ with m edges and vertex degree sequence $d_1 \geq d_2 \geq \cdots \geq d_n$. Topological indices represent numerical graph invariants used to correlate the physical and chemical properties of molecules [19]. Numerous degree-based graph invariants have been proposed in the literature. Among them, the first Zagreb index is one of the most widely investigated molecular structure descriptors. It is defined as [9]

$$M_1(G) = \sum_{i=1}^{n} d_i^2.$$

The forgotten topological index, defined as the sum of cubes of vertex degrees, is a variant of the first Zagreb index. It is given by [6]

$$F(G) = \sum_{i=1}^{n} d_i^3.$$

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Another degree-based graph invariant, the degree deviation, is proposed as the quantitative topological characterization of irregularity of a graph. It is defined as [14]

$$S(G)=\sum_{i=1}^{n}\left|d_i-\frac{2m}{n}\right|.$$

Let $A(G)$ be the adjacency matrix of the graph G with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ [5]. The resolvent energy of G is defined as [8]

$$ER(G)=\sum_{i=1}^{n}\frac{1}{n-\lambda_i}. \tag{1}$$

For a detailed investigation of $ER(G)$, we refer the reader to recent studies [1, 2, 20, 21].

Let $D(G)=\operatorname{diag}(d_1,d_2,\ldots,d_n)$ denote the diagonal degree matrix of G. The Laplacian matrix of G is defined as $L(G)=D(G)-A(G)$ [12]. Eigenvalues of $L(G)$, $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} > \mu_n = 0$, represent Laplacian eigenvalues of G [7]. Analogous to the definition of resolvent energy given by (1), the Laplacian resolvent energy was introduced in [4] as

$$RL(G)=\sum_{i=1}^{n}\frac{1}{n+1-\mu_i}. \tag{2}$$

Basic properties and several bounds of $RL(G)$ can be found [3, 11, 15, 17, 22].

Let K_n and $K_{1,n-1}$ denote the complete graph and the star graph on n vertices, respectively. In the following, we recall two previously established lower bounds for $RL(G)$. In [15], Palacios determined a lower bound on $RL(G)$ as follows:

$$RL(G)\geq\frac{1}{n-d_1}+\sum_{i=2}^{n-1}\frac{1}{n+1-d_i}+\frac{1}{n+2-d_n}. \tag{3}$$

Equality in (3) is attained by the star graph $K_{1,n-1}$.

Matejić et al. [11] reported that

$$RL(G)\geq\frac{1}{n+1}+\frac{1}{n-d_1}+\frac{(n-2)^2}{(n+1)(n-2)-2m+d_1+1}. \tag{4}$$

Equality in (4) holds if and only if $G\cong K_n$, or $G\cong K_{1,n-1}$.

In this study, by providing lower bounds for the quantity $\sum_{i=2}^{n-1}\frac{1}{n+1-d_i}$ given in (3), we aim to derive new lower bounds for $RL(G)$ involving the vertex degrees d_1, d_2, d_{n-1} and d_n as well as the degree-based graph invariants, specifically the degree deviation, the first Zagreb index and the forgotten topological index.

2 Preliminaries

In this section, we recall several analytical inequalities for real number sequences that are fundamental to deriving our main results.

Lemma 1. [18] *Let $a = (a_i)$, $i = 1,2,\ldots n$, be a real number sequence such that $0 < q \leq a_i \leq Q < +\infty$. Then the following inequality holds*

$$n\sum_{i=1}^{n} a_i^2 - \left(\sum_{i=1}^{n} a_i\right)^2 \geq \frac{n}{2}(Q-q)^2. \tag{5}$$

Lemma 2. [10, 13] *Let $p = (p_i)$, $i = 1,2,\ldots,n$, be a sequence of positive real numbers, and let $a = (a_i)$, $b = (b_i),\ldots,c = (c_i)$ $i = 1,2,\ldots,n$, be r sequences of non-negative real numbers such that all monotonic in the same sense. Then it holds that*

$$\left(\sum_{i=1}^{n} p_i\right)^{r-1} \sum_{i=1}^{n} p_i a_i b_i \ldots c_i \geq \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i \ldots \sum_{i=1}^{n} p_i c_i \tag{6}$$

with equality if and only if $r-1$ sequences are constant.

Lemma 3. [16] *Let $x = (x_i)$ and $a = (a_i)$, $i = 1,2,\ldots n$, be two sequences of positive real numbers. Then, for any $r \geq 0$ holds*

$$\sum_{i=1}^{n} \frac{x_i^{r+1}}{a_i^r} \geq \frac{\left(\sum\limits_{i=1}^{n} x_i\right)^{r+1}}{\left(\sum\limits_{i=1}^{n} a_i\right)^{r}} \tag{7}$$

with equality if and only if $r = 0$, or $\frac{x_1}{a_1} = \frac{x_2}{a_2} = \cdots = \frac{x_n}{a_n}$.

3 Main Results

In the following theorem, we establish a lower bound for $RL(G)$ in terms the number of vertices n, the number of edges m and the vertex degrees d_1, d_2, d_{n-1} and d_n.

Theorem 1. *Let G be a simple connected graph of order $n \geq 3$ with m edges and the vertex degree sequence $d_1 \geq d_2 \geq \cdots \geq d_n$. Then*

$$\begin{aligned} RL(G) \quad \geq \quad & \frac{1}{n-d_1} + + \frac{1}{n+2-d_n} + \frac{(n-2)^2}{(n+1)(n-2)-(2m-d_1-d_n)} \\ & + \frac{\left(\sqrt{n+1-d_2} - \sqrt{n+1-d_{n-1}}\right)^2}{2(n+1-d_2)(n+1-d_{n-1})}. \end{aligned} \tag{8}$$

Equality in (8) *occurs when $G = K_{1,n-1}$.*

Proof. For $i = 2,\ldots,n-1$, the inequality (5) can be written as:

$$(n-2)\sum_{i=2}^{n-1} a_i^2 - \left(\sum_{i=2}^{n-1} a_i\right)^2 \geq \frac{n-2}{2}(Q-q)^2.$$

Setting $a_i = 1/\sqrt{n+1-d_i}, i = 2,\ldots,n-1$, $Q = 1/\sqrt{n+1-d_2}$ and $q = 1/\sqrt{n+1-d_{n-1}}$ in the above, we obtain

$$\begin{aligned}
&(n-2)\sum_{i=2}^{n-1}\frac{1}{n+1-d_i} - \left(\sum_{i=2}^{n-1}\frac{1}{\sqrt{n+1-d_i}}\right)^2 \\
\geq\ & \frac{n-2}{2}\left(\frac{1}{\sqrt{n+1-d_2}} - \frac{1}{\sqrt{n+1-d_{n-1}}}\right)^2,
\end{aligned}$$

that is,

$$\begin{aligned}
&(n-2)\sum_{i=2}^{n-1}\frac{1}{n+1-d_i} - \left(\sum_{i=2}^{n-1}\frac{1}{\sqrt{n+1-d_i}}\right)^2 \\
\geq\ & \frac{(n-2)\left(\sqrt{n+1-d_2} - \sqrt{n+1-d_{n-1}}\right)^2}{2\,(n+1-d_2)\,(n+1-d_{n-1})}.
\end{aligned} \tag{9}$$

Analogously, for $i = 2,\ldots,n-1$, the inequality (6) is given by

$$\left(\sum_{i=2}^{n-1} p_i\right)^{r-1}\sum_{i=2}^{n-1} p_i a_i b_i \ldots c_i \geq \sum_{i=2}^{n-1} p_i a_i \sum_{i=2}^{n-1} p_i b_i \ldots \sum_{i=2}^{n-1} p_i c_i.$$

For $r = 3$, $p_i = 1/\sqrt{n+1-d_i}$ and $a_i = b_i = c_i = \sqrt{n+1-d_i}, i = 2,\ldots,n-1$, this inequality takes the form

$$\left(\sum_{i=2}^{n-1}\frac{1}{\sqrt{n+1-d_i}}\right)^2\sum_{i=2}^{n-1}(n+1-d_i) \geq \left(\sum_{i=2}^{n-1} 1\right)^3,$$

that is,

$$\left(\sum_{i=2}^{n-1}\frac{1}{\sqrt{n+1-d_i}}\right)^2 \geq \frac{(n-2)^3}{(n+1)\,(n-2) - (2m-d_1-d_n)}. \tag{10}$$

Then by (9) and (10), we have

$$\begin{aligned}
&\sum_{i=2}^{n-1}\frac{1}{n+1-d_i} \\
\geq\ & \frac{(n-2)^2}{(n+1)\,(n-2) - (2m-d_1-d_n)} + \frac{\left(\sqrt{n+1-d_2} - \sqrt{n+1-d_{n-1}}\right)^2}{2\,(n+1-d_2)\,(n+1-d_{n-1})}.
\end{aligned} \tag{11}$$

From (3) and (11), we arrive at (8). Equality in (10) occurs when $d_2 = \cdots = d_{n-1}$. Equality in (3) is attained by the star graph $K_{1,n-1}$. Therefore, we conclude that equality in (8) occurs when $G = K_{1,n-1}$. $\square$

Remark 1. *Obviously* $\frac{\left(\sqrt{n+1-d_2}-\sqrt{n+1-d_{n-1}}\right)^2}{2(n+1-d_2)(n+1-d_{n-1})} \geq 0$. *Therefore the lower bound* (8) *is at least as strong as* (4)*, and strictly stronger than* (4) *for every non-star tree graphs due to their structural property* $d_n = 1$.

Remark 2. *Note that for $G = K_{1,n-1}$, $n \geq 3$, the lower bounds* (4) *and* (8) *coincide, and are equal to*

$$RL(K_{1,n-1}) = 1 + \frac{n-2}{n} + \frac{1}{n+1}.$$

Let us consider the path graph P_5 and the tree graph T with 6 *vertices and degree sequence* $(3,3,1,1,1,1)$. *For these graphs, $RL(P_5) \approx 1.277$ and $RL(T) \approx 1.289$, respectively. Rounded to three decimal places, for the path graph P_5, the lower bound* (4) *yields $RL(P_5) \geq 1.192$ while the lower bound* (8) *yields $RL(P_5) \geq 1.194$. Similarly, for the* 6*-vertex tree T, the lower bound* (4) *gives $RL(T) \geq 1.203$ whereas the lower bound* (8) *gives $RL(T) \geq 1.208$. Although the lower bounds* (4) *and* (8) *yield very close numerical values to each other, these examples explicitly validate that our lower bound* (8) *provides a consistent alternative and is strictly stronger than the previous lower bound* (4) *for non-star tree graphs.*

The following theorem gives a lower bound for $RL(G)$ in terms of degree deviation $S(G)$, the first Zagreb index $M_1(G)$ and the forgotten topological index $F(G)$.

Theorem 2. *Let G be a simple connected graph of order $n \geq 3$ and size m, and vertex degree sequence $d_1 \geq d_2 \geq \cdots \geq d_n$, with at least one $d_i \neq \frac{2m}{n}$, $i \in \{2, \ldots, n-1\}$. Then*

$$RL(G) \geq \frac{1}{n-d_1} + \frac{1}{n+2-d_n} + \frac{(S(G) - d_1 + d_n)^2}{I(G)} \tag{12}$$

where

$$\begin{aligned} I(G) &= \left(n+1+\frac{4m}{n}\right)\left(M_1(G) - d_1^2 - d_n^2\right) - \left(F(G) - d_1^3 - d_n^3\right) \\ &\quad - \left(\frac{4m}{n}(n+1) + \frac{4m^2}{n^2}\right)(2m - d_1 - d_n) + \frac{4m^2}{n^2}(n+1)(n-2). \end{aligned}$$

Equality is attained by the star $G = K_{1,n-1}$.

Proof. The inequality (7) can also be considered in the following form

$$\sum_{i=2}^{n-1} \frac{x_i^{r+1}}{a_i^r} \geq \frac{\left(\sum\limits_{i=2}^{n-1} x_i\right)^{r+1}}{\left(\sum\limits_{i=2}^{n-1} a_i\right)^r}.$$

For $r = 1$, $x_i = \left|d_i - \frac{2m}{n}\right|$, $a_i = \frac{1}{n+1-d_i}$, $i = 2, \ldots, n-1$, this inequality transforms to

$$\sum_{i=2}^{n-1} \left|d_i - \frac{2m}{n}\right|^2 (n+1-d_i) \geq \frac{\left(\sum\limits_{i=2}^{n-1} \left|d_i - \frac{2m}{n}\right|\right)^2}{\sum\limits_{i=2}^{n-1} \frac{1}{n+1-d_i}}. \tag{13}$$

On the other hand, we have

$$\begin{aligned}
\sum_{i=2}^{n-1}\left(d_i-\frac{2m}{n}\right)^2(n+1-d_i) &= \left(n+1+\frac{4m}{n}\right)\sum_{i=2}^{n-1}d_i^2-\sum_{i=2}^{n-1}d_i^3 \\
&\quad -\left(\frac{4m}{n}(n+1)+\frac{4m^2}{n^2}\right)\sum_{i=2}^{n-1}d_i+\left(\frac{4m^2}{n^2}(n+1)\right)\sum_{i=2}^{n-1}1 \\
&= \left(n+1+\frac{4m}{n}\right)\left(M_1(G)-d_1^2-d_n^2\right)-\left(F(G)-d_1^3-d_n^3\right) \\
&\quad -\left(\frac{4m}{n}(n+1)+\frac{4m^2}{n^2}\right)(2m-d_1-d_n) \\
&\quad +\frac{4m^2}{n^2}(n+1)(n-2). \qquad (14)
\end{aligned}$$

Thus by (13) and (14), we get

$$\sum_{i=2}^{n-1}\frac{1}{n+1-d_i}\geq\frac{\left(S(G)-\left|d_1-\frac{2m}{n}\right|-\left|d_n-\frac{2m}{n}\right|\right)^2}{I(G)},$$

that is,

$$\sum_{i=2}^{n-1}\frac{1}{n+1-d_i}\geq\frac{(S(G)-d_1+d_n)^2}{I(G)}. \qquad (15)$$

From (3) and (15), the inequality (12) follows. Equality in (13) occurs when $d_2=\cdots=d_{n-1}$. Equality in (3) is attained by the star graph $K_{1,n-1}$. Accordingly, equality in (12) occurs when $G=K_{1,n-1}$. □

4 Conclusions

In this paper, we find some new lower bounds for $RL(G)$ involving the vertex degrees d_1, d_2, d_{n-1}, d_n and various degree-based graph invariants. Although the lower bounds (4) and (8) yield very close numerical values to each other, the examples explicitly validate that our lower bound (8) provides a consistent alternative and is strictly stronger than the previous lower bound (4) for non-star tree graphs.

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