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Remark on delta and reverse inverse degree indices

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Abstract: Let $G = (V, E)$, $V = \{v_1, v_2, \ldots, v_n\}$, be a simple graph of order n and size m, with vertex–degree sequence $\Delta = d_1 \ge d_2 \ge \cdots \ge d_n = \delta > 0$. and let $s_1 \ge s_2 \ge \cdots \ge s_n$, $s_i = d_i - \delta + 1$ and $c_1 \le c_2 \le \cdots \le c_n$, $c_i = \Delta - d_i + 1$, be two vertex–degree like sequences. By analogy with the inverse degree graph invariant, $ID(G) = \sum_{i=1}^{n} \frac{1}{d_i}$, the delta inverse degree and reverse inverse degree indices are defined, respectively, as $\delta ID(G) = \sum_{i=1}^{n} \frac{1}{s_i}$ and $RID(G) = \sum_{i=1}^{n} \frac{1}{c_i}$. In this paper we determine sharp bounds on $\delta ID(G)$ and $RID(G)$ and the extremal graphs are characterized.
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1 Introduction

Let $G = (V, E)$, $V = \{v_1, v_2, \ldots, v_n\}$, be a simple graph with n vertices, m edges with vertex–degree sequence $\Delta = d_1 \ge d_2 \ge \cdots \ge d_n = \delta > 0$, $d_i = d(v_i)$. If vertices v_i and v_j are adjacent in G, we write $i \sim j$, otherwise we write $i \nsim j$. With $\overline{G}$ we denote the complement of graph G.

In graph theory, a graph invariant is property of the graph that is preserved by isomorphisms. Obviously the simplest invariants are the number of vertices and the number of edges. The graph invariants that assume only numerical values are usually referred to as topological indices in chemical graph theory. Hundreds of various topological indices have been introduced in mathematical chemistry literature in order to describe physical and chemical properties of molecules. Many of them are defined as simple functions of the degree sequence of (molecular) graph. Most of the degree–based topological indices are viewed as the contributions of pairs of adjacent vertices. These type of indices are known as the bond incident degree (BID in short) indices [17]. Various mathematical properties of topological indices have been investigated, as well.

A wide class of vertex–degree–based topological indices (see, for example, [1, 3, 9]), are defined as

$$TI_f(G) = \sum_{i \sim j} (f(d_i) + f(d_j)) = \sum_{i=1}^{n} d_i f(d_i). \tag{1.1}$$

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When $f(x) = \frac{1}{x^2}$ we get the inverse degree index, $ID(G)$, introduced in [8] as

$$ID(G) = \sum_{i \sim j} \left(\frac{1}{d_i^2} + \frac{1}{d_j^2} \right) = \sum_{i=1}^{n} \frac{1}{d_i}.$$

A new vertex–degree–like sequence $s_1 \geq s_2 \geq \cdots \geq s_n$, $s_i = d_i - \delta + 1$, $s_1 = \Delta - \delta + 1$, $s_n = 1$, was introduced in [11]. By analogy with $ID(G)$, delta inverse degree index, $\delta ID(G)$, can be defined as

$$\delta ID(G) = \sum_{i=1}^{n} \frac{1}{s_i}. \tag{1.2}$$

Also, according to the additive representation of $ID(G)$, delta inverse vertex degree beta index, $\delta ID^{\beta}(G)$, can be defined as

$$\delta ID^{\beta}(G) = \sum_{i \sim j} \left(\frac{1}{s_i^2} + \frac{1}{s_j^2} \right) = \sum_{i=1}^{n} \frac{d_i}{s_i^2}. \tag{1.3}$$

One can easily see that for the graphs with the property $\delta = 1$, holds

$$ID(G) = \delta ID(G) = \delta ID^{\beta}(G).$$

Of course, in general this is not true.

Similarly as in [11], in [5] (see also [6, 7, 16]) a new vertex–degree–like sequence $c_1 \leq c_2 \leq \cdots \leq c_n$, was introduced as $c_i = \Delta - d_i + 1$, $c_1 = 1$, $c_n = \Delta - \delta + 1$. By analogy with indices $\delta ID(G)$ and $\delta ID^{\beta}(G)$, reverse inverse degree index, $RID(G)$, can be defined as

$$RID(G) = \sum_{i=1}^{n} \frac{1}{c_i} \tag{1.4}$$

and reverse inverse vertex degree beta index, $RID^{\beta}(G)$, as

$$RID^{\beta}(G) = \sum_{i \sim j} \left(\frac{1}{c_i^2} + \frac{1}{c_j^2} \right) = \sum_{i=1}^{n} \frac{d_i}{c_i^2}. \tag{1.5}$$

In general, topological indices $ID(G)$, $RID(G)$ and $RID^{\beta}(G)$ are different.

The concept of coindices was introduced in [4](see also [2]). In this case the sum runs over the edges of the complement of G. In a view of (1.1), the corresponding coindex of G can be defined as [1]

$$\overline{TI}_f(G) = \sum_{i \nsim j} (f(d_i) + f(d_j)) = \sum_{i=1}^{n} (n - 1 - d_i) f(d_i). \tag{1.6}$$

Based on (1.6), topological coindices corresponding to indices $\delta ID^{\beta}(G)$ and $RID^{\beta}(G)$ are defined as

$$\overline{\delta ID}^{\beta}(G) = \sum_{i \nsim j} \left(\frac{1}{s_i^2} + \frac{1}{s_j^2} \right) = \sum_{i=1}^{n} (n - 1 - d_i) \frac{1}{s_i^2} \tag{1.7}$$

and

$$\overline{RID}^{\beta}(G) = \sum_{i \nsim j} \left(\frac{1}{c_i^2} + \frac{1}{c_j^2} \right) = \sum_{i=1}^{n} (n-1-d_i) \frac{1}{c_i^2}. \tag{1.8}$$

In this paper we determine sharp bounds for the above introduced indices/coindices and the extremal graphs are characterized, considering numerical inequalities present in the literature.

2 Main results

At the beginning let us recall a couple analytical inequalities for real number sequences which will be used in the proofs of theorems.

Lemma 2.1. [15] *Let $x = (x_i)$ and $a = (a_i)$, $i = 1, 2, \ldots, n$, be positive real number sequences. Then for any $r \geq 0$ holds*

$$\sum_{i=1}^{n} \frac{x_i^{r+1}}{a_i^r} \geq \frac{\left(\sum\limits_{i=1}^{n} x_i \right)^{r+1}}{\left(\sum\limits_{i=1}^{n} a_i \right)^{r}}. \tag{2.1}$$

Equality holds if and only if $\frac{x_1}{a_1} = \frac{x_2}{a_2} = \cdots = \frac{x_n}{a_n}$.

The inequality (2.1) is known in the literature as Radon's inequality. Here it is given in its original form. But, it is not difficult to verify that it is valid for any real r such that $r \leq -1$ or $r \geq 0$, and when $-1 \leq r \leq 0$ the opposite inequality holds. Equality holds if and only if $r = -1$, or $r = 0$, or $\frac{x_1}{a_1} = \frac{x_2}{a_2} = \cdots = \frac{x_n}{a_n}$.

Lemma 2.2. [10] *Let $a = (a_i)$, $i = 1, 2, \ldots, n$, be a real number sequence with the property $0 < r \leq a_i \leq R < +\infty$. Then*

$$\sum_{i=1}^{n} a_i \sum_{i=1}^{n} \frac{1}{a_i} \leq n^2 \left(1 + \alpha(n) \frac{(R-r)^2}{rR} \right), \tag{2.2}$$

where

$$\alpha(n) = \frac{1}{4} \left(1 - \frac{(-1)^{n+1} + 1}{2n^2} \right).$$

In the next theorem we determine a lower bound on reverse inverse degree index, $RID(G)$, in terms of basic graph parameters n, m, Δ and δ.

Theorem 2.1. *Let G be a connected graph with $n \geq 2$ vertices and m edges. Then*

$$RID(G) \geq 1 + \frac{1}{\Delta - \delta + 1} + \frac{(n-2)^2}{n(\Delta+1) - 2m - (\Delta - \delta + 2)}. \tag{2.3}$$

Equality holds if and only if $d_2 = d_3 = \cdots = d_{n-1}$.

Proof. Let $a_2, a_3, \ldots, a_{n-1}$ be positive real numbers. According to the arithmetic–harmonic mean inequality (see, for example, [13]), we have that

$$\sum_{i=2}^{n-1} a_i \sum_{i=2}^{n-1} \frac{1}{a_i} \geq (n-2)^2. \tag{2.4}$$

Equality holds if and only if $a_2 = a_3 = \cdots = a_{n-1}$.

For $a_i = c_i$, $i = 2, 3, \ldots, n-1$, this inequality becomes

$$\sum_{i=2}^{n-1} c_i \sum_{i=2}^{n-1} \frac{1}{c_i} \geq (n-2)^2, \tag{2.5}$$

i.e.

$$\left(\sum_{i=1}^{n} c_i - c_1 - c_n\right)\left(\sum_{i=1}^{n} \frac{1}{c_i} - \frac{1}{c_1} - \frac{1}{c_n}\right) \geq (n-2)^2.$$

Since

$$\sum_{i=1}^{n} c_i = n(\Delta+1) - 2m, \quad c_1 = 1, \quad c_n = \Delta - \delta + 1,$$

from the above it follows

$$(n(\Delta+1) - 2m - (\Delta - \delta + 2))\left(RID(G) - 1 - \frac{1}{\Delta - \delta + 1}\right) \geq (n-2)^2,$$

from which we get (2.3).

Equality in (2.5) holds if and only if $c_2 = c_3 = \cdots = c_{n-1}$, which implies that equality in (2.3) holds if and only if $d_2 = d_3 = \cdots = d_{n-1}$. □

Corollary 2.1. *Let G be a connected graph with $n \geq 2$ vertices, m edges such that $\delta = 1$. Then*

$$RID(G) \geq 1 + \frac{1}{\Delta} + \frac{(n-2)^2}{(n-1)(\Delta+1) - 2m}.$$

Equality holds if and only if $d_2 = d_3 = \cdots = d_{n-1}$.

The proof of the next theorem is analogous to that of Theorem 2.1, hence omitted.

Theorem 2.2. *Let G be a connected graph with $n \geq 2$ vertices and m edges. Then*

$$\delta ID(G) \geq 1 + \frac{1}{\Delta - \delta + 1} + \frac{(n-2)^2}{2m - n(\delta - 1) - (\Delta - \delta + 2)}. \tag{2.6}$$

Equality holds if and only if $d_2 = d_3 = \cdots = d_{n-1}$.

Corollary 2.2. *Let G be a connected graph with $n \geq 2$ vertices, m edges such that $\delta = 1$. Then*

$$\delta ID(G) = ID(G) \geq 1 + \frac{1}{\Delta} + \frac{(n-2)^2}{2m - (\Delta+1)}. \tag{2.7}$$

Equality holds if and only if $d_2 = d_3 = \cdots = d_{n-1}$.

Proof. When $\delta = 1$, we have that $s_i = d_i$, for every $i = 1, 2, \dots, n$. Having this in mind, the inequality (2.7) directly follows from (2.6). $\square$

Since

$$\begin{aligned} c_i &= \Delta - d_i + 1 = \Delta - (n-1) + (n-1) - d_i + 1 \\ &= d_i(\overline{G}) - \delta(\overline{G}) + 1 = s_i(\overline{G}) = \overline{s}_i, \end{aligned} \tag{2.8}$$

for every $i = 1, 2, \dots, n$, according to (2.3) and (2.6) in the next theorem we obtain the inequality of Nordhaus–Gaddum type (see [14]) for $RID(G)$.

Theorem 2.3. *Let G be a connected graph with $n \geq 2$ vertices and m edges. Then*

$$\begin{aligned} RID(G) + RID(\overline{G}) \geq 2 &+ \frac{2}{\Delta - \delta + 1} \\ &+ \frac{(\Delta - \delta + 2)(n-2)^3}{(n(\Delta+1) - 2m - (\Delta - \delta + 2))(2m - n(\delta - 1) - (\Delta - \delta + 2))}. \end{aligned}$$

Equality holds if and only if $d_2 = d_3 = \cdots = d_{n-1}$.

Proof. From (2.8) it follows

$$RID(\overline{G}) = \sum_{i=1}^{n} \frac{1}{\overline{c}_i} = \sum_{i=1}^{n} \frac{1}{s_i} = \delta ID(G),$$

and therefore

$$RID(G) + RID(\overline{G}) = RID(G) + \delta ID(G).$$

From the above identity and (2.3) and (2.6) we get the desired result. $\square$

Corollary 2.3. *Let G be a connected graph with $n \geq 2$ vertices, m edges with the property $\delta = 1$. Then*

$$\begin{aligned} RID(G) + RID(\overline{G}) &= RID(G) + ID(G) \\ &\geq 2 + \frac{2}{\Delta} + \frac{(n-2)^3(\Delta+1)}{((n-1)(\Delta+1) - 2m)(2m - (\Delta+1))}. \end{aligned}$$

Equality holds if and only if $d_2 = d_3 = \cdots = d_{n-1}$.

In the next theorem we determine an upper bound for $RID(G)$ in terms of parameters n, m, Δ and δ.

Theorem 2.4. *Let G be a connected graph with $n \geq 2$ vertices and m edges. Then*

$$RID(G) \leq \frac{2m - n(\delta - 1)}{\Delta - \delta + 1}. \tag{2.9}$$

Equality holds if and only if $d_i \in \{\Delta, \delta\}$ for every $i = 1, 2, \dots, n$.

Proof. Since

$$\min_i\{c_i\} = c_1 = 1 \quad \text{and} \quad \max_i\{c_i\} = c_n = \Delta - \delta + 1,$$

for every $i = 1, 2, \ldots, n$, it follows

$$(\Delta - \delta + 1 - c_i)\left(1 - \frac{1}{c_i}\right) \geq 0,$$

that is

$$(\Delta - \delta + 1)\frac{1}{c_i} \leq d_i - (\delta - 1). \tag{2.10}$$

After summation of the above inequality over i, $i = 1, 2, \ldots, n$, we get

$$(\Delta - \delta + 1)RID(G) \leq 2m - n(\delta - 1), \tag{2.11}$$

from which we get (2.9).

Equality in (2.10) holds if and only if $c_i \in \{\Delta - \delta + 1, 1\}$ for every i, $i = 1, 2, \ldots, n$, and therefore equality in (2.9) holds if and only if $d_i \in \{\Delta, \delta\}$ for every $i = 1, 2, \ldots, n$. □

Corollary 2.4. *Let G be a connected graph with $n \geq 2$ vertices and m edges. Then wev have*

$$RID(G) \leq \frac{n^2(\Delta - \delta + 2)^2}{4(n(\Delta + 1) - 2m)(\Delta - \delta + 1)}. \tag{2.12}$$

Equality holds if and only if G is a regular graph or $\Delta = d_1 = \cdots = d_{\frac{n}{2}} > d_{\frac{n}{2}+1} = \cdots = d_n = \delta$, for even n.

Proof. According to (2.11) we have that

$$(\Delta - \delta + 1)RID(G) \leq 2m - n(\delta - 1) = n(\Delta - \delta + 2) - (n(\Delta + 1) - 2m),$$

that is

$$(n(\Delta + 1) - 2m) + (\Delta - \delta + 1)RID(G) \leq n(\Delta - \delta + 2).$$

By the arithmetic–geometric mean inequality (see e.g [13]) we have that

$$2\sqrt{(\Delta - \delta + 1)(n(\Delta + 1) - 2m)RID(G)} \leq n(\Delta - \delta + 2),$$

from which we arrive at (2.12). □

Corollary 2.5. *Let G be a connected graph with $n \geq 2$ vertices, m edges with the property $\delta = 1$. Then*

$$RID(G) \leq \frac{2m}{\Delta}.$$

Equality holds if and only if $d_i \in \{\Delta, 1\}$ for every $i = 1, 2, \ldots, n$.

By a similar procedure as in case of Theorem 2.4, we get the following result.

Theorem 2.5. *Let G be a connected graph with $n \geq 2$ vertices and m edges. Then*

$$\delta ID(G) \leq \frac{n(\Delta + 1) - 2m}{\Delta - \delta + 1}.$$

Equality holds if and only if $d_i \in \{\Delta, \delta\}$ for every $i = 1, 2, \ldots, n$.

The proof of the next corollary of Theorem 2.5 is similar to that of Corollary 2.4, therefore omitted.

Corollary 2.6. *Let G be a connected graph with $n \geq 2$ vertices and m edges. Then we have*

$$\delta ID(G) \leq \frac{n^2(\Delta+\delta+2)^2}{4(2m-n(\delta-1))(\Delta-\delta+1)}\,. \tag{2.13}$$

Equality holds if and only if G is a regular graph or $\Delta = d_1 = \cdots = d_{\frac{n}{2}} > d_{\frac{n}{2}+1} = \cdots = d_n = \delta$, for even n.

Corollary 2.7. *Let G be a connected graph with $n \geq 2$ vertices, m edges with the property $\delta = 1$. Then*

$$\delta ID(G) = ID(G) \leq \frac{n(\Delta+1)-2m}{\Delta}.$$

Equality holds if and only if $d_i \in \{\Delta, 1\}$ for every $i = 1, 2, \ldots, n$.

Corollary 2.8. *Let T be a tree with $n \geq 2$ vertices. Then*

$$\delta ID(T) = ID(T) \leq n - \frac{n-2}{\Delta}.$$

Equality holds if and only if $d_i \in \{\Delta, 1\}$ for every $i = 1, 2, \ldots, n$.

The above inequality was proven in [12].

In a similar way as in Theorem 2.3, we get the following result.

Theorem 2.6. *Let G be a connected graph with $n \geq 2$ vertices. Then*

$$RID(G) + RID(\overline{G}) \leq \frac{n(\Delta-\delta+2)}{\Delta-\delta+1}.$$

Equality holds if and only if $d_i \in \{\Delta, \delta\}$ for every $i = 1, 2, \ldots, n$.

Corollary 2.9. *Let G be a connected graph with $n \geq 2$ vertices, m edges with the property $\delta = 1$. Then*

$$RID(G) + RID(\overline{G}) \leq \frac{n(\Delta+1)}{\Delta}.$$

Equality holds if and only if $d_i \in \{\Delta, 1\}$ for every $i = 1, 2, \ldots, n$.

Theorem 2.7. *Let G be a connected graph with $n \geq 2$ vertices and m edges. Then*

$$RID(G) \leq \frac{n^2}{n(\Delta+1)-2m}\left(1+\alpha(n)\frac{(\Delta-\delta)^2}{\Delta-\delta+1}\right), \tag{2.14}$$

where

$$\alpha(n) = \frac{1}{4}\left(1-\frac{(-1)^{n+1}+1}{2n^2}\right).$$

Equality holds if and only if G is a regular graph.

Proof. For $a_i = c_i$, $i = 1, 2, \dots, n$, $R = \Delta - \delta + 1$, $r = 1$, the inequality (2.2) becomes

$$\sum_{i=1}^{n} c_i \sum_{i=1}^{n} \frac{1}{c_i} \le n^2 \left(1 + \alpha(n) \frac{(\Delta - \delta)^2}{\Delta - \delta + 1}\right) . \tag{2.15}$$

From the above we have that

$$(n(\Delta + 1) - 2m) RID(G) \le n^2 \left(1 + \alpha(n) \frac{(\Delta - \delta)^2}{\Delta - \delta + 1}\right) ,$$

which yields the inequality (2.14). The equality in (2.15) holds if and only if $c_1 = c_2 = \cdots = c_n$, that is if and only if $d_1 = d_2 = \cdots = d_n$. This implies that G is a regular graph. □

Remark 2.1. *Since $\alpha(n) \le \frac{1}{4}$, for any n, the inequality (2.14) is stronger than (2.12) whenever n is odd.*

The proof of the next theorem is fully analogous to that of Theorem 2.7, thus omitted.

Theorem 2.8. *Let G be a connected graph with $n \ge 2$ vertices and m edges. Then*

$$\delta ID(G) \le \frac{n^2}{2m - n(\delta - 1)} \left(1 + \alpha(n) \frac{(\Delta - \delta)^2}{\Delta - \delta + 1}\right) .$$

Equality holds if and only if G is a regular graph.

Corollary 2.10. *Let G be a connected graph with $n \ge 2$ vertices and m edges. Then we have*

$$RID(G) + RID(\overline{G}) \le \frac{n^3(\Delta - \delta + 2)}{(2m - n(\delta - 1))(n(\Delta + 1) - 2m)} \left(1 + \alpha(n) \frac{(\Delta - \delta)^2}{\Delta - \delta + 1}\right) .$$

Equality holds if and only if G is a regular graph.

In the next theorem we determine a lower bound for $RID(T)$, where T is a tree, in terms of parameters n and Δ.

Theorem 2.9. *Let T be a tree with $n \ge 3$ vertices. Then*

$$RID(T) \ge 1 + \frac{2}{\Delta} + \frac{(n - 3)^2}{n(\Delta - 1) + 1 - 2\Delta}. \tag{2.16}$$

Equality holds if and only if $T \cong P_n$, or $T \cong K_{1,n-1}$.

Proof. The inequality (2.4) can be considered in the form

$$\sum_{i=2}^{n-2} a_i \sum_{i=2}^{n-2} \frac{1}{a_i} \ge (n - 3)^2.$$

For $a_i = c_i$, $i = 2, 3, \dots, n - 2$, this inequality becomes

$$\sum_{i=2}^{n-2} c_i \sum_{i=2}^{n-2} \frac{1}{c_i} \ge (n - 3)^2,$$

i.e.

$$\left(\sum_{i=1}^{n} c_i - c_1 - c_{n-1} - c_n\right)\left(\sum_{i=1}^{n}\frac{1}{c_i} - \frac{1}{c_1} - \frac{1}{c_{n-1}} - \frac{1}{c_n}\right) \geq (n-3)^2. \tag{2.17}$$

Since T is a tree, it has at least two vertices of degree 1, therefore $d_{n-1} = d_n = 1$, i.e. $c_{n-1} = c_n = \Delta$. Now (2.17) becomes

$$(n(\Delta+1) - 2(n-1) - 1 - 2\Delta)\left(RID(T) - 1 - \frac{2}{\Delta}\right) \geq (n-3)^2,$$

from which we arrive at (2.16).

Equality in (2.17) holds if and only if $c_2 = c_3 = \cdots = c_{n-2}$, that is if and only if $d_2 = d_3 = \cdots = d_{n-2}$. Since T is a tree, $d_{n-1} = d_n = 1$, we have that

$$\Delta + (n-3)d_2 = 2(n-2).$$

If $d_2 = 1$, then $\Delta = n-1$. If $d_2 = 2$, then $\Delta = d_1 = d_2 = \cdots = d_{n-2} = 2$. If $d_2 \geq 3$, then $\Delta \geq 3$, and therefore

$$3(n-2) \leq 2(n-2),$$

which does not hold for any $n \geq 3$. Finally, equality in (2.16) holds if and only if $T \cong P_n$, or $T \cong K_{1,n-1}$, $n \geq 3$. □

In the following theorem we establish relation between $RID(G)$ and $\overline{RID}^{\beta}(G)$.

Theorem 2.10. *Let G be a connected graph with $n \geq 3$ vertices and m edges. Then*

$$\overline{RID}^{\beta}(G) - RID(G) \geq \frac{(n-\Delta-2)n^3}{(n(\Delta+1)-2m)^2}. \tag{2.18}$$

Equality holds if and only if G is a regular graph.

Proof. According to (1.8) it holds

$$\begin{aligned}\overline{RID}^{\beta}(G) &= \sum_{i=1}^{n}(n-1-d_i)\frac{1}{c_i^2} = \sum_{i=1}^{n}(\Delta - d_i + 1 - \Delta + n - 1 - 1)\frac{1}{c_i^2} \\ &= \sum_{i=1}^{n}(c_i + n - \Delta - 2)\frac{1}{c_i^2} = \sum_{i=1}^{n}\frac{1}{c_i} + (n-\Delta-2)\sum_{i=1}^{n}\frac{1}{c_i^2} \\ &= RID(G) + (n-\Delta-2)\sum_{i=1}^{n}\frac{1}{c_i^2}.\end{aligned} \tag{2.19}$$

On the other hand, for $r = 2$, $x_i = 1$, $a_i = c_i$, $i = 1, 2, \ldots, n$, the inequality (2.1) transforms into

$$\sum_{i=1}^{n}\frac{1^3}{c_i^2} \geq \frac{\left(\sum_{i=1}^{n} 1\right)^3}{\left(\sum_{i=1}^{n} c_i\right)^2} = \frac{n^3}{(n(\Delta+1)-2m)^2}. \tag{2.20}$$

From the above and (2.19) we arrive at (2.18).

Equality in (2.20) holds if and only if $\frac{1}{c_1} = \frac{1}{c_2} = \cdots = \frac{1}{c_n}$, therefore equality in (2.18) holds if and only if $d_1 = d_2 = \cdots = d_n$, i.e. if and only if G is a regular graph. □

By a similar procedure as in the case of Theorem 2.10, we get the following result.

Theorem 2.11. *Let G be a graph with $n \geq 3$ vertices and m edges. Then*

$$\delta ID(G) + \overline{\delta ID}^{\beta}(G) \geq \frac{(n-\delta)n^3}{(2m - n(\delta-1))^2}.$$

Equality holds if and only if G is a regular graph.

Corollary 2.11. *Let G be a connected graph of order $n \geq 2$ and size m, such that $\delta = 1$. Then we have*

$$\delta ID(G) + \overline{\delta ID}^{\beta}(G) = ID(G) + \overline{ID}(G) \geq \frac{(n-1)n^3}{4m^2}.$$

Equality holds if and only if $G \cong \frac{n}{2} K_2$, n is even.

3 Conclusions

We have obtained upper and lower bounds on the reverse inverse degree, $RID(G)$, and delta inverse degree, $\delta ID(G)$, topological indices in terms of basic graph parameters, that is number of vertices, n, number of edges, m, and maximal and minimal vertex degrees, Δ and δ. Extremal graphs were determined as well. Then we prove the Nordhous–Gaddum type inequalities for these indices. At the end we considered a relationship between $RID(G)$ and $\delta D(G)$ and the corresponding coindices.

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