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Nested Functions of Type Supertrigonometric and Superhyperbolic via Mittag-Leffler Functions

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Abstract: In this paper nested functions of type supertrigonometric and superhyperbolic are introduced and their properties are determined. Using these functions some classes of fractional differential equations are solved.

Keywords: supertrigonometric and superhyperbolic functions, Mittag-Leffler functions, fractional differential equations

1 Introduction

The Mittag-Leffler function is the special function of the form

$$E_{\alpha}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)},\ \ \alpha,z\in\mathbb{C},\ \Re(\alpha)>0,$$

and it is introduced in [13, 14] by Swedish mathematician Gosta Magnus Mittag-Leffler. Later, Wiman [20] introduced the two-parameter Mittag–Leffler function $E_{\alpha,\beta}(z)$, which is given by

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)},\ \alpha,\beta,z\in\mathbb{C},\ \Re(\alpha)>0,\ \Re(\beta)>0.$$

It is proved in [7] (see also [11]) that

$$\mathscr{L}(E_{\alpha}(\pm\lambda t^{\alpha}))(s)=\frac{s^{\alpha-1}}{s^{\alpha}\mp\lambda} \tag{1}$$

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and

$$\mathscr{L}(t^{\beta-1}E_{\alpha,\beta}(\pm\lambda t^{\alpha}))(s) = \frac{s^{\alpha-\beta}}{s^{\alpha} \mp \lambda}, \tag{2}$$

where $\Re(s) > 0,\ \lambda \in \mathbb{C},\ |\lambda s^{-\alpha}| < 1$. With the development of the fractional calculus, the interest of scientists in the study of Mittag-Leffler functions grew [6–9]. Solutions of some linear fractional differential equations may be expressed in terms of the Mittag–Leffler function [3, 4]. During the last two decades, the interest in Mittag-Leffler functions and Mittag-Leffler type functions have grown, especially among engineers and scientists due to their huge application in several applied problems, such as fluid flow, diffusion-like transport, electrical networks, probability and statistical distribution theory [5, 15, 18]. They are particularly interesting the supersine and supercosine functions based on the Mittag-Leffler function introduced in [3] (see also [10, 19]).

In this paper we introduce nested functions of type supertrigonometric $T_{pj\alpha},\ T_{pj\alpha,\beta}$ and superhyperbolic $H_{pj\alpha},\ H_{pj\alpha,\beta}$ as the special case of Mittag-Leffler functions in order to solve some classes of fractional differential equations. We suggest the word „nested", because with the finite derivative they become functions in the same class. We also gave some properties of these functions.

2 Preliminaries

In this subsection we recall some important results regarding the functions T_{pj} and H_{pj} and the fractional calculus.

2.1 Notions

We use the following notation: $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ for the sets of positive integers, real and complex numbers, respectively; $\Re(z)$ is the real part of the complex number z. A Laplace transform of a function f is denoted by $\mathscr{L}(f(t))(s) = \int_0^{+\infty} f(t)e^{st}dt,\ s \in \mathbb{C}$, and the gamma function is denoted $\Gamma(z) = \int_0^{+\infty} t^{z-1}e^{-t}dt$, $\Re(z) > 0$.

2.2 Functions T_{pj} and H_{pj}

The functions $T_{pj}, H_{pj} : \mathbb{R} \to \mathbb{R},\ j = 0, 1, 2, \cdots, p-1,\ p \in \mathbb{N}$, are defined as follows [1]:

$$T_{pj}(t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{pn+j}}{(pn+j)!}, \quad H_{pj}(t) = \sum_{n=0}^{\infty} \frac{t^{pn+j}}{(pn+j)!}.$$

Theorem 2.1. *[1] For each $t \in \mathbb{R}$, we have*

$$T_{30}^3(t) - T_{31}^3(t) + T_{32}^3(t) + 3T_{30}(t)T_{31}(t)T_{32}(t) = 1$$
$$H_{30}^3(t) + H_{31}^3(t) + H_{32}^3(t) - 3H_{30}(t)H_{31}(t)H_{32}(t) = 1.$$

Theorem 2.2. *[1] Let $w^3 = -1, w \neq -1$. Then for each $t \in \mathbb{R}$, we have*

$$\begin{aligned} \exp(-t) &= T_{30}(t) - T_{31}(t) + T_{32}(t) \\ \exp(wt) &= T_{30}(t) + wT_{31}(t) + w^2 T_{32}(t) \\ \exp(-w^2 t) &= T_{30}(t) - w^2 T_{31}(t) - wT_{32}(t), \end{aligned}$$

and

$$\begin{aligned} T_{30}(t) &= \frac{\exp(-t) + \exp(wt) + \exp(-w^2 t)}{3} \\ T_{31}(t) &= \frac{-\exp(-t) - w^2 \exp(wt) + w\exp(-w^2 t)}{3} \\ T_{32}(t) &= \frac{\exp(-t) - w\exp(wt) + w^2 \exp(-w^2 t)}{3}. \end{aligned}$$

Theorem 2.3. *Let $\lambda^3 = 1, \lambda \neq 1$. Then for each $t \in \mathbb{R}$, we have*

$$\begin{aligned} \exp(t) &= H_{30}(t) + H_{31}(t) + H_{32}(t) \\ \exp(\lambda t) &= H_{30}(t) + \lambda H_{31}(t) + \lambda^2 H_{32}(t) \\ \exp(\lambda^2 t) &= H_{30}(t) + \lambda^2 H_{31}(t) + \lambda H_{32}(t), \end{aligned}$$

and

$$\begin{aligned} H_{30}(t) &= \frac{\exp(t) + \exp(\lambda t) + \exp(\lambda^2 t)}{3} \\ H_{31}(t) &= \frac{\exp(t) + \lambda^2 \exp(\lambda t) + \lambda \exp(\lambda^2 t)}{3} \\ H_{32}(t) &= \frac{\exp(t) + \lambda \exp(\lambda t) + \lambda^2 \exp(\lambda^2 t)}{3}. \end{aligned}$$

2.3 Fractional derivative

The Riemann-Liouville fractional derivative of order α is defined by [12, 16]

$${}_aD^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{y(u)}{(t-u)^{\alpha-n+1}} du, \quad n-1 < \alpha \leq n.$$

In this paper we consider the case $a = 0$ and we denote it by D^{α}. The Laplace transform of the fractional derivative is given by [12, 16, 17]

$$\mathscr{L}(D^{\alpha} y(t))(s) = s^{\alpha} \mathscr{L}(y(t))(s) - \sum_{j=0}^{n-1} s^j D^{\alpha-j-1} y(0), \quad n-1 < \alpha \leq n.$$

According to the terminology used by Miller and Ross, [12], the fractional differential equation

$$D^{\sigma_n} y(t) + \sum_{j=1}^{n-1} p_j(t) D^{\sigma_{n-j}} y(t) + p_n(t) y(t) = h(t), D^{\sigma_j - 1} y(0) = b_j, \ j = 1, 2, \ldots, n, \tag{3}$$

is a sequential fractional differential equation where

$$D^{\sigma_j} = D^{\alpha_j} D^{\alpha_{j-1}} \cdots D^{\alpha_1}, \;\; D^{\sigma_j - 1} = D^{\alpha_j - 1} D^{\alpha_{j-1}} \cdots D^{\alpha_1},$$

$$\sigma_j = \sum_{i=0}^{j} \alpha_i, \; j = 1, \ldots, n, \; 0 < \alpha_i < 1, \;\; i = 1, \ldots, n.$$

The Laplace transform of the sequential fractional partial derivative $D^{\sigma_n} y(t)$ is given by [17]

$$\mathscr{L}(D^{\sigma_n} y(t))(s) = s^{\sigma_n} \mathscr{L}(y(t))(s) - \sum_{j=0}^{n-1} s^{\sigma_n - \sigma_{n-j}} D^{\sigma_{n-j} - 1} y(0). \tag{4}$$

3 Supertrigonometric and superhyperbolic functions via Mittag-Leffler functions

Following [18] we introduce supertrigonometric superhyperbolic functions.

Definition 3.1. *Let $T_{pj\alpha} : \mathbb{R} \to \mathbb{R}, j = 0, 1, 2, \cdots, p-1, \; p \in \mathbb{N}, \; \alpha \in \mathbb{C}, \; \Re(\alpha) > 0$. We say that the function $T_{pj\alpha}$ is type pre-supertrigonometric with p element if*

$$\text{presuper}\, T_{pj\alpha}(t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{pn+j}}{\Gamma((pn+j)\alpha + 1)}.$$

Definition 3.2. *Let $T_{pj\alpha} : \mathbb{R} \to \mathbb{R}, j = 0, 1, 2, \cdots, p-1, \; p \in \mathbb{N}, \; \alpha \in \mathbb{C}, \; \Re(\alpha) > 0$. We say that the function $T_{pj\alpha}$ is type supertrigonometric with p element if*

$$\text{super}\, T_{pj\alpha}(t^{\alpha}) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{(pn+j)\alpha}}{\Gamma((pn+j)\alpha + 1)}.$$

Firstly, we consider the case when $p = 3$. Using Theorem 2.2 we obtain connections between pre-supertrigonometric and supertrigonometric with Mittag-Leffler functions.

Theorem 3.3. *Let $w^3 = -1, w \neq -1, \; \alpha \in \mathbb{C}, \; \Re(\alpha) > 0$. Then for each $t \in \mathbb{R}$, we have*

$$\begin{aligned}
\text{presuper}\, T_{30\alpha}(t) &= \frac{E_\alpha(-t) + E_\alpha(wt) + E_\alpha(-w^2 t)}{3} \\
\text{presuper}\, T_{31\alpha}(t) &= \frac{-E_\alpha(-t) - w^2 E_\alpha(wt) + w E_\alpha(-w^2 t)}{3} \\
\text{presuper}\, T_{32\alpha}(t) &= \frac{E_\alpha(-t) - w E_\alpha(wt) + w^2 E_\alpha(-w^2 t)}{3}
\end{aligned}$$

and

$$\begin{aligned}
\text{super}\, T_{30\alpha}(t^{\alpha}) &= \frac{E_\alpha(-t^{\alpha}) + E_\alpha(wt^{\alpha}) + E_\alpha(-w^2 t^{\alpha})}{3} \\
\text{super}\, T_{31\alpha}(t^{\alpha}) &= \frac{-E_\alpha(-t^{\alpha}) - w^2 E_\alpha(wt^{\alpha}) + w E_\alpha(-w^2 t^{\alpha})}{3} \\
\text{super}\, T_{32\alpha}(t^{\alpha}) &= \frac{E_\alpha(-t^{\alpha}) - w E_\alpha(wt^{\alpha}) + w^2 E_\alpha(-w^2 t^{\alpha})}{3}
\end{aligned}$$

Theorem 3.4. *Let* $w^3 = -1, w \neq -1,\ \alpha \in \mathbb{C},\ \Re(\alpha) > 0$. *Then for each* $t \in \mathbb{R}$, *we have*

$$\begin{aligned}&(\mathrm{presuper}\,T_{30\alpha}(t))^3 - (\mathrm{presuper}\,T_{31\alpha}(t))^3 + (\mathrm{presuper}\,T_{32\alpha}(t))^3 + \\ &3\,\mathrm{presuper}\,T_{30\alpha}(t)\,\mathrm{presuper}\,T_{31\alpha}(t)\,\mathrm{presuper}\,T_{32\alpha}(t) \\ &= E_\alpha(-t)E_\alpha(wt)E_\alpha(-w^2 t).\end{aligned}$$

and

$$\begin{aligned}&(\mathrm{super}\,T_{30\alpha}(t^\alpha))^3 - (\mathrm{super}\,T_{31\alpha}(t^\alpha))^3 + (\mathrm{super}\,T_{32\alpha}(t^\alpha))^3 + \\ &3\,\mathrm{super}\,T_{30\alpha}(t^\alpha)\,\mathrm{super}\,T_{31\alpha}(t^\alpha)\,\mathrm{super}\,T_{32\alpha}(t^\alpha) \\ &= E_\alpha(-t^\alpha)E_\alpha(wt^\alpha)E_\alpha(-w^2 t^\alpha).\end{aligned}$$

Proof. The proof follows from Theorem 2.1 and Theorem 3.3. □

Using Theorem 3.3 we obtain the following assertion.

Theorem 3.5. *Let* $w^p = -1, w \neq -1$, $p \geq 3$ *is a prime number and* $\alpha \in \mathbb{C}$, $\Re(\alpha) > 0$. *Then for each* $t \in \mathbb{R}$ *and* $j = 0, 1, \ldots, p-1$, *we have*

$$\mathrm{presuper}\,T_{pj\alpha}(t) = \frac{(-1)^{j+1}}{p} \sum_{i=0}^{p-1} (-1)^{ij} \omega^{p-ij} E_\alpha((-1)^{i+1}\omega^i t)$$

and

$$\mathrm{super}\,T_{pj\alpha}(t^\alpha) = \frac{(-1)^{j+1}}{p} \sum_{i=0}^{p-1} (-1)^{ij} \omega^{p-ij} E_\alpha((-1)^{i+1}\omega^i t^\alpha). \tag{5}$$

Definition 3.6. *Let* $H_{pj\alpha} : \mathbb{R} \to \mathbb{R},\ j = 0, 1, 2, \cdots, p-1,\ p \in \mathbb{N},\ \alpha \in \mathbb{C},\ \Re(\alpha) > 0$. *We say that the function* $H_{pj\alpha}$ *is type pre-superhyperbolic with p element if*

$$\mathrm{presuper}\,H_{pj\alpha}(t) = \sum_{n=0}^{\infty} \frac{t^{pn+j}}{\Gamma((pn+j)\alpha+1)}.$$

Definition 3.7. *Let* $H_{pj\alpha} : \mathbb{R} \to \mathbb{R},\ j = 0, 1, 2, \cdots, p-1,\ p \in \mathbb{N},\ \alpha \in \mathbb{C},\ \Re(\alpha) > 0$. *We say that the function* $H_{pj\alpha}$ *is type superhyperbolic with p element if*

$$\mathrm{super}\,H_{pj\alpha}(t^\alpha) = \sum_{n=0}^{\infty} \frac{t^{(pn+j)\alpha}}{\Gamma((pn+j)\alpha+1)}.$$

Firstly we consider a case when $p = 3$. Using Theorem 2.3 we obtain connections between pre-superhyperbolic and superhyperbolic with Mittag-Leffler functions.

Theorem 3.8. *Let* $\lambda^3 = 1, \lambda \neq 1,\ \alpha \in \mathbb{C},\ \Re(\alpha) > 0$. *Then for each* $t \in \mathbb{R}$, *we have*

$$\begin{aligned}\mathrm{presuper}\,H_{30\alpha}(t) &= \frac{E_\alpha(t) + E_\alpha(\lambda t) + E_\alpha(\lambda^2 t)}{3} \\ \mathrm{presuper}\,H_{31\alpha}(t) &= \frac{E_\alpha(t) + \lambda^2 E_\alpha(\lambda t) + \lambda E_\alpha(\lambda^2 t)}{3} \\ \mathrm{presuper}\,H_{32\alpha}(t) &= \frac{E_\alpha(t) + \lambda E_\alpha(\lambda t) + \lambda^2 E_\alpha(\lambda^2 t)}{3}\end{aligned}$$

and

$$\begin{aligned}
\text{super}H_{30\alpha}(t^{\alpha}) &= \frac{E_{\alpha}(t^{\alpha})+E_{\alpha}(\lambda t^{\alpha})+E_{\alpha}(\lambda^2 t^{\alpha})}{3} \\
\text{super}H_{31\alpha}(t^{\alpha}) &= \frac{E_{\alpha}(t^{\alpha})+\lambda^2 E_{\alpha}(\lambda t^{\alpha})+\lambda E_{\alpha}(\lambda^2 t^{\alpha})}{3} \\
\text{super}H_{32\alpha}(t^{\alpha}) &= \frac{E_{\alpha}(t^{\alpha})+\lambda E_{\alpha}(\lambda t^{\alpha})+\lambda^2 E_{\alpha}(\lambda^2 t^{\alpha})}{3}.
\end{aligned}$$

Theorem 3.9. *Let $\lambda^3 = 1, \lambda \neq 1$, $\alpha \in \mathbb{C}$, $\Re(\alpha) > 0$. Then for each $t \in \mathbb{R}$, we have*

$$\begin{aligned}
&(\text{presuper}H_{30\alpha}(t))^3 + (\text{presuper}H_{31\alpha}(t))^3 + (\text{presuper}H_{32\alpha}(t))^3 - \\
&3\,\text{presuper}H_{30\alpha}(t)\,\text{presuper}H_{31\alpha}(t)\,\text{presuper}H_{32\alpha}(t) \\
&= E_{\alpha}(t)E_{\alpha}(\lambda t)E_{\alpha}(\lambda^2 t)
\end{aligned}$$

and

$$\begin{aligned}
&(\text{super}H_{30\alpha}(t^{\alpha}))^3 + (\text{super}H_{31\alpha}(t^{\alpha}))^3 + (\text{super}H_{32\alpha}(t^{\alpha}))^3 - \\
&3\,\text{super}H_{30\alpha}(t^{\alpha})\,\text{super}H_{31\alpha}(t^{\alpha})\,\text{super}H_{32\alpha}(t^{\alpha}) \\
&= E_{\alpha}(t^{\alpha})E_{\alpha}(\lambda t^{\alpha})E_{\alpha}(\lambda^2 t^{\alpha}).
\end{aligned}$$

Proof. The proof follows from Theorem 3.8 and Theorem 2.1. □

Theorem 3.10. *Let $\lambda^p = 1, \lambda \neq 1$, $p \geq 3$ is a prime number and $\alpha \in \mathbb{C}$, $\Re(\alpha) > 0$. Then for each $t \in \mathbb{R}$ and $j = 0, 1, \ldots, p-1$ we have*

$$\text{presuper}H_{pj\alpha}(t) = \frac{1}{p}\sum_{i=0}^{p-1} \lambda^{p-ij} E_{\alpha}(\lambda^i t),$$

$$\text{super}H_{pj\alpha}(t^{\alpha}) = \frac{1}{p}\sum_{i=0}^{p-1} \lambda^{p-ij} E_{\alpha}(\lambda^i t^{\alpha}).$$

Theorem 3.11. *Let $p \geq 3$ be prime number, $\alpha, s \in \mathbb{C}$, $\Re(\alpha) > 0$, $\Re(s) > 0$ and $|s^{-\alpha}| < 1$. Then we have*

$$\mathscr{L}(\text{super}T_{pj\alpha}(t^{\alpha}))(s) = \frac{s^{(p-j)\alpha-1}}{s^{p\alpha}+1}, \quad j = 0, 1, ..., p-1 \tag{6}$$

and

$$\mathscr{L}(\text{super}H_{pj\alpha}(t^{\alpha}))(s) = \frac{s^{(p-j)\alpha-1}}{s^{p\alpha}-1}, \quad j = 0, 1, ..., p-1. \tag{7}$$

Proof. We will prove (6), since the proof for (7) is similar. If we apply the Laplace transform on (5), using (1), we obtain

$$\begin{aligned}
\mathscr{L}(\text{super}T_{pj\alpha}(t^{\alpha}))(s) &= \frac{(-1)^{j+1}}{p}\sum_{i=0}^{p-1}(-1)^{ij}\omega^{p-ij}\frac{s^{\alpha-1}}{s^{\alpha}+(-\omega)^i} \\
&= \frac{(-1)^{j+1}}{p}\frac{s^{\alpha-1}}{s^{p\alpha}+1}\sum_{i=0}^{p-1}(-1)^{ij}\omega^{p-ij}\sum_{k=0}^{p-1}(-1)^k s^{(p-1-k)\alpha}(-\omega)^{ik}.
\end{aligned}$$

Since $\omega^p = -1$, $\omega \neq -1$, then $\sum_{k=0}^{p-1}(-1)^k\omega^{p-1-k} = 0$, so we have

$$\frac{(-1)^{j+1}}{p}\sum_{i=0}^{p-1}(-1)^{ij}\omega^{p-ij}\sum_{k=0}^{p-1}(-1)^k s^{(p-1-k)\alpha}(-\omega)^{ik} = \frac{(-1)^{j+p+2}}{p}\sum_{i=0}^{p-1}(-1)^{ij}\omega^{p-ij}(-\omega)^{i(p-1)}$$
$$+s^{\alpha}\frac{(-1)^{j+p}}{p}\sum_{i=0}^{p-1}(-1)^{ij}\omega^{p-ij}(-\omega)^{i(p-2)}+\ldots+s^{(p-j-1)\alpha}\frac{(-1)^{2j+1}}{p}\sum_{i=0}^{p-1}\omega^p+\ldots$$
$$+s^{(p-1)\alpha}\frac{(-1)^{j+1}}{p}\sum_{i=0}^{p-1}(-1)^{ij}\omega^{p-ij} = 0+s^{\alpha}\cdot 0+\ldots+s^{(p-j-1)\alpha}+\ldots+0.$$

This completes the proof. □

Theorem 3.12. *Let $p \geq 3$ be a prime number, $a > 0$, $\alpha, s \in \mathbb{C}$, $\Re(\alpha) > 0$, $\Re(s) > 0$ and $|as^{-\alpha}| < 1$. Then we have*

$$\mathscr{L}(\text{super}\,T_{pj\alpha}(at^{\alpha}))(s) = \frac{s^{(p-j)\alpha-1}a^j}{s^{p\alpha}+a^p}, \quad j = 0,1,...,p-1 \tag{8}$$

and

$$\mathscr{L}(\text{super}\,H_{pj\alpha}(at^{\alpha}))(s) = \frac{s^{(p-j)\alpha-1}a^j}{s^{p\alpha}-a^p}, \quad j = 0,1,...,p-1. \tag{9}$$

Proof. Follows from Theorem 3.11. □

Definition 3.13. *Let $T_{pj\alpha,\beta} : \mathbb{R} \to \mathbb{R}$, $j = 0,1,2,\cdots,p-1$, $p \in \mathbb{N}$, $\alpha, \beta \in \mathbb{C}$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$. We say that $T_{pj\alpha,\beta}$ is type pre-supertrigonometric with p element if*

$$\text{presuper}\,T_{pj\alpha,\beta}(t) = \sum_{n=0}^{\infty}\frac{(-1)^n t^{pn+j}}{\Gamma((pn+j)\alpha+\beta)}.$$

Definition 3.14. *Let $T_{pj\alpha,\beta} : \mathbb{R} \to \mathbb{R}$, $j = 0,1,2,\cdots,p-1$, $p \in \mathbb{N}$, $\alpha, \beta \in \mathbb{C}$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$. We say that $T_{pj\alpha,\beta}$ is type supertrigonometric with p element if*

$$\text{super}\,T_{pj\alpha,\beta}(t^{\alpha}) = \sum_{n=0}^{\infty}\frac{(-1)^n t^{(pn+j)\alpha+\beta-1}}{\Gamma((pn+j)\alpha+\beta)}$$

Firstly, we consider the case $p = 3$.

Theorem 3.15. *Let $w^3 = -1, w \neq -1$, $\alpha, \beta \in \mathbb{C}$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$. Then for each $t \in \mathbb{R}$, we have*

$$\begin{aligned}
\text{presuper}\,T_{30\alpha,\beta}(t) &= \frac{E_{\alpha,\beta}(-t)+E_{\alpha,\beta}(wt)+E_{\alpha,\beta}(-w^2t)}{3}\\
\text{presuper}\,T_{31\alpha,\beta}(t) &= \frac{-E_{\alpha,\beta}(-t)-w^2E_{\alpha,\beta}(wt)+wE_{\alpha,\beta}(-w^2t)}{3}\\
\text{presuper}\,T_{32\alpha,\beta}(t) &= \frac{E_{\alpha,\beta}(-t)-wE_{\alpha,\beta}(wt)+w^2E_{\alpha,\beta}(-w^2t)}{3}
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{super} T_{30\alpha,\beta}(t^{\alpha}) &= t^{\beta-1}\frac{E_{\alpha,\beta}(-t^{\alpha})+E_{\alpha,\beta}(wt^{\alpha})+E_{\alpha,\beta}(-w^2t^{\alpha})}{3}\\
\operatorname{super} T_{31\alpha,\beta}(t^{\alpha}) &= t^{\beta-1}\frac{-E_{\alpha,\beta}(-t^{\alpha})-w^2E_{\alpha,\beta}(wt^{\alpha})+wE_{\alpha,\beta}(-w^2t^{\alpha})}{3}\\
\operatorname{super} T_{32\alpha,\beta}(t^{\alpha}) &= t^{\beta-1}\frac{E_{\alpha,\beta}(-t^{\alpha})-wE_{\alpha,\beta}(wt^{\alpha})+w^2E_{\alpha,\beta}(-w^2t^{\alpha})}{3}.
\end{aligned}$$

Theorem 3.16. *Let $w^3=-1, w\neq -1$, $\alpha,\beta\in\mathbb{C}$, $\Re(\alpha)>0$, $\Re(\beta)>0$. Then for each $t\in\mathbb{R}$ we have*

$$\begin{aligned}
&(\operatorname{presuper} T_{30\alpha,\beta}(t))^3-(\operatorname{presuper} T_{31\alpha,\beta}(t))^3+(\operatorname{presuper} T_{32\alpha,\beta}(t))^3+\\
&3\operatorname{presuper} T_{30\alpha,\beta}(t)\operatorname{presuper} T_{31\alpha,\beta}(t)\operatorname{presuper} T_{32\alpha,\beta}(t)\\
&=E_{\alpha,\beta}(-t)E_{\alpha,\beta}(wt)E_{\alpha,\beta}(-w^2t)
\end{aligned}$$

and

$$\begin{aligned}
&(\operatorname{super} T_{30\alpha,\beta}(t^{\alpha}))^3-(\operatorname{super} T_{31\alpha,\beta}(t^{\alpha}))^3+(\operatorname{super} T_{32\alpha,\beta}(t^{\alpha}))^3+\\
&3\operatorname{super} T_{30\alpha,\beta}(t^{\alpha})\operatorname{super} T_{31\alpha,\beta}(t^{\alpha})\operatorname{super} T_{32\alpha,\beta}(t^{\alpha})\\
&=E_{\alpha,\beta}(-t^{\alpha})E_{\alpha,\beta}(wt^{\alpha})E_{\alpha,\beta}(-w^2t^{\alpha}).
\end{aligned}$$

In general case, we have the following assertion.

Theorem 3.17. *Let $w^p=-1, w\neq -1$, $p\geq 3$ be a prime number and $\alpha,\beta\in\mathbb{C}$, $\Re(\alpha)>0, \Re(\beta)>0$. Then for each $t\in\mathbb{R}$ and $j=0,1,\ldots,p-1$ we have*

$$\begin{aligned}
\operatorname{presuper} T_{pj\alpha,\beta}(t) &= \frac{(-1)^{j+1}}{p}\sum_{i=0}^{p-1}(-1)^{ij}\omega^{p-ij}E_{\alpha,\beta}((-1)^{i+1}\omega^i t)\\
\operatorname{super} T_{pj\alpha,\beta}(t^{\alpha}) &= \frac{(-1)^{j+1}}{p}\sum_{i=0}^{p-1}(-1)^{ij}\omega^{p-ij}t^{\beta-1}E_{\alpha,\beta}((-1)^{i+1}\omega^i t^{\alpha}).
\end{aligned}$$

Definition 3.18. *Let $H_{pj\alpha,\beta}:\mathbb{R}\to\mathbb{R}$, $j=0,1,2,\cdots,p-1$, $p\in\mathbb{N}$, $\alpha,\beta\in\mathbb{C}$, $\Re(\alpha)>0, \Re(\beta)>0$. We say that $H_{pj\alpha,\beta}$ is type pre-superhyperbolic with p element if*

$$\operatorname{presuper} H_{pj\alpha,\beta}(t)=\sum_{n=0}^{\infty}\frac{t^{pn+j}}{\Gamma((pn+j)\alpha+\beta)}.$$

Definition 3.19. *Let $H_{pj\alpha,\beta}:\mathbb{R}\to\mathbb{R}$, $j=0,1,2,\cdots,p-1$, $p\in\mathbb{N}$. We say that $H_{pj\alpha,\beta}$ is type superhyperbolic with p element if*

$$\operatorname{super} H_{pj\alpha,\beta}(t^{\alpha})=\sum_{n=0}^{\infty}\frac{t^{(pn+j)\alpha+\beta-1}}{\Gamma((pn+j)\alpha+\beta)}.$$

Theorem 3.20. *Let $\lambda^3 = 1, \lambda \neq 1,\ \alpha, \beta \in \mathbb{C},\ \Re(\alpha) > 0,\ \Re(\beta) > 0$. Then for each $t \in \mathbb{R}$, we have*

$$\begin{aligned}
\text{presuper}H_{30\alpha,\beta}(t) &= \frac{E_{\alpha,\beta}(t) + E_{\alpha,\beta}(\lambda t) + E_{\alpha,\beta}(\lambda^2 t)}{3} \\
\text{presuper}H_{31\alpha,\beta}(t) &= \frac{E_{\alpha,\beta}(t) + \lambda^2 E_{\alpha,\beta}(\lambda t) + \lambda E_{\alpha,\beta}(\lambda^2 t)}{3} \\
\text{presuper}H_{32\alpha,\beta}(t) &= \frac{E_{\alpha,\beta}(t) + \lambda E_{\alpha,\beta}(\lambda t) + \lambda^2 E_{\alpha,\beta}(\lambda^2 t)}{3}
\end{aligned}$$

and

$$\begin{aligned}
\text{super}H_{30\alpha,\beta}(t^\alpha) &= t^{\beta-1}\frac{E_{\alpha,\beta}(t^\alpha) + E_{\alpha,\beta}(\lambda t^\alpha) + E_{\alpha,\beta}(\lambda^2 t^\alpha)}{3} \\
\text{super}H_{31\alpha,\beta}(t^\alpha) &= t^{\beta-1}\frac{E_{\alpha,\beta}(t^\alpha) + \lambda^2 E_{\alpha,\beta}(\lambda t^\alpha) + \lambda E_{\alpha,\beta}(\lambda^2 t^\alpha)}{3} \\
\text{super}H_{32\alpha,\beta}(t^\alpha) &= t^{\beta-1}\frac{E_{\alpha,\beta}(t^\alpha) + \lambda E_{\alpha,\beta}(\lambda t^\alpha) + \lambda^2 E_{\alpha,\beta}(\lambda^2 t^\alpha)}{3}
\end{aligned}$$

Theorem 3.21. *Let $\lambda^3 = 1, \lambda \neq 1,\ \alpha, \beta \in \mathbb{C},\ \Re(\alpha) > 0,\ \Re(\beta) > 0$. Then for each $t \in \mathbb{R}$, we have*

$$\begin{aligned}
&(\text{presuper}H_{30\alpha,\beta}(t))^3 + (\text{presuper}H_{31\alpha,\beta}(t))^3 + (\text{presuper}H_{32\alpha,\beta}(t))^3 - \\
&3\,\text{presuper}H_{30\alpha,\beta}(t)\,\text{presuper}H_{31\alpha,\beta}(t)\,\text{presuper}H_{32\alpha,\beta}(t) \\
&= E_{\alpha,\beta}(t)E_{\alpha,\beta}(\lambda t)E_{\alpha,\beta}(\lambda^2 t)
\end{aligned}$$

and

$$\begin{aligned}
&(\text{super}H_{30\alpha,\beta}(t^\alpha))^3 + (\text{super}H_{31\alpha,\beta}(t^\alpha))^3 + (\text{super}H_{32\alpha,\beta}(t^\alpha))^3 - \\
&3\,\text{super}H_{30\alpha,\beta}(t^\alpha)\,\text{super}H_{31\alpha,\beta}(t^\alpha)\,\text{super}H_{32\alpha,\beta}(t^\alpha) \\
&= E_{\alpha,\beta}(t^\alpha)E_{\alpha,\beta}(\lambda t^\alpha)E_{\alpha,\beta}(\lambda^2 t^\alpha).
\end{aligned}$$

Theorem 3.22. *Let $\lambda^p = 1, \lambda \neq 1,\ p \geq 3$ is a prime and $\alpha, \beta \in \mathbb{C},\ \Re(\alpha) > 0,\ \Re(\beta) > 0$. Then for each $t \in \mathbb{R}$ and $j = 0, 1, \ldots, p-1$ we have*

$$\text{presuper}H_{pj\alpha,\beta}(t) = \frac{1}{p}\sum_{i=0}^{p-1} \lambda^{p-ij} E_{\alpha,\beta}(\lambda^i t)$$

$$\text{super}H_{pj\alpha,\beta}(t^\alpha) = \frac{1}{p}\sum_{i=0}^{p-1} \lambda^{p-ij} t^{\beta-1} E_{\alpha,\beta}(\lambda^i t^\alpha)$$

Theorem 3.23. *Let $p \geq 3$ be a prime number, $\alpha, \beta, s \in \mathbb{C},\ \Re(\alpha) > 0,\ \Re(\beta) > 0,\ \Re(s) > 0$ and $|s^{-\alpha}| < 1$. Then we have*

$$\mathscr{L}(\text{super}T_{pj\alpha,\beta}(t^\alpha))(s) = \frac{s^{(p-j)\alpha-\beta}}{s^{p\alpha}+1}, \quad j = 0, 1, ..., p-1 \tag{10}$$

and

$$\mathscr{L}(\text{super}H_{pj\alpha,\beta}(t^{\alpha}))(s) = \frac{s^{(p-j)\alpha-\beta}}{s^{p\alpha}-1}, \quad j = 0,1,...,p-1. \tag{11}$$

Proof. The proof follows from Theorem 3.11 and (2). □

Theorem 3.24. *Let $p \geq 3$ be a prime number, $\alpha, \beta, s \in \mathbb{C}$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(s) > 0$, $a > 0$ and $|as^{-\alpha}| < 1$. Then we have*

$$\mathscr{L}(\text{super}T_{pj\alpha,\beta}(at^{\alpha}))(s) = \frac{s^{(p-j)\alpha-\beta}a^{j}}{s^{p\alpha}+a^{p}}, \quad j = 0,1,...,p-1 \tag{12}$$

and

$$\mathscr{L}(\text{super}H_{pj\alpha,\beta}(at^{\alpha}))(s) = \frac{s^{(p-j)\alpha-\beta}a^{j}}{s^{p\alpha}-a^{p}}, \quad j = 0,1,...,p-1. \tag{13}$$

4 Solving fractional differential equations using supertrigonometric and superhyperbolic functions

This section is dedicated to present a new method for solving fractional differential equations. This new method is based on using the supertrigonometric and superhyperbolic functions for solving one class of fractional differential equations. This method can be extended for many types of fractional differential equations, [2].

Theorem 4.1. *Let $p \geq 3$ be a prime number $a, b \in \mathbb{R}$, $a > 0$ and $0 < \alpha \leq 1$. The solution of the sequential fractional differential equation*

$$D^{p\alpha}y(t) + ay(t) = b \tag{14}$$

with initial conditions $D^{\alpha(p-j)-1}y(0) = b_j$, $j = 1,\ldots,p$ is

$$y(t) = \sum_{j=0}^{p-1} \frac{b_{p-j}}{\sqrt[p]{a^{j}}} T_{pj\alpha,\alpha}(\sqrt[p]{a}t^{\alpha}) + \frac{b}{a} - \frac{b}{a} T_{p0\alpha}(\sqrt[p]{a}t^{\alpha}).$$

Proof. If we apply the Laplace transform on (14), $\Re(s) > 0$, $|as^{-\alpha}| < 1$ using (4) (we put in (3) $\alpha_i = \alpha$) we obtain

$$(s^{p\alpha} + a)\mathscr{L}(y(t))(s) = \sum_{j=0}^{p-1} s^{j\alpha} b_{j+1} + \frac{b}{s}.$$

Then

$$\begin{aligned}\mathscr{L}(y(t))(s) &= \sum_{j=0}^{p-1} b_{p-j} \frac{s^{\alpha(p-j)-\alpha}}{s^{\alpha p}+a} + \frac{b}{s(s^{\alpha p}+1)} \\ &= \sum_{j=0}^{p-1} b_{p-j} \frac{s^{\alpha(p-j)-\alpha}\sqrt[p]{a^{j}}}{\sqrt[p]{a^{j}}(s^{\alpha p}+a)} + \frac{b}{as} - \frac{bs^{\alpha p-1}}{s(s^{\alpha p}+a)}.\end{aligned}$$

If we apply the inverse Laplace transform on the previous equality, using (12) and (8) we obtain the assertion. □

Theorem 4.2. *Let $p \geq 3$ be a prime number $a, b \in \mathbb{R}$, $a > 0$ and $0 < \alpha \leq 1$. The solution of the sequential fractional differential equation*

$$D^{p\alpha} y(t) - a y(t) = b \tag{15}$$

with initial conditions $D^{\alpha(p-j)-1} y(0) = b_j$, $j = 1, \ldots, p$ is

$$y(t) = \sum_{j=0}^{p-1} \frac{b_{p-j}}{\sqrt[p]{a^j}} H_{pj\alpha,\alpha}(\sqrt[p]{a} t^{\alpha}) + \frac{b}{a} - \frac{b}{a} H_{p0\alpha}(\sqrt[p]{a} t^{\alpha}).$$

Proof. The proof is similar like those in Theorem 4.1. □

References

[1] A. H. ANSARI, X.L. LIU, V. N. MISHRA, *On Mittag-Leffler function and beyond*, Nonlinear Sci. Lett. A, vol.8, 2 (2017), 187-199.

[2] A. B. ANTONEVICH, E. V. KUZMINA, *On Generalized Solutions of Some Differential Equations with Singular Coefficients*, Sci. Pub. State Univ. Novi Pazar, Ser A: Appl. Math. Inform. Mech. vol. 14, 1 (2022), 1-11.

[3] R.L. BAGLEY, *The initial value problem for fractional order differential equations with constant coefficients,* Air Force Institute of Technology Report, AFIT-TR-EN-88-1, 1988.

[4] G. DATTOLI, K. GORSKA, A. HORZELA, S. LICCIARDI, R.M. PIDATELLA, *Comments on the properties of Mittag-Leffler function*, Eur. Phys. J. Spec. Top. vol. 226, 16–18 (2017), 3427–3443.

[5] A. ERDELYI, W. MAGNUS, F. OBERHETTINGER, F.G. TRICOMI, *Higher transcendental functions*, vol. 3, McGraw-Hill, New York, NY, 1955.

[6] R. GORENFLO, A.A. KILBAS, F. MAINARDI, S.V. ROGOSIN, *Mittag-Leffler Functions, related topics and applications*, vol. 2. Springer, Berlin, 2014.

[7] R. GORENFLO, A. A. KILBAS, F.MAINARDI, S. ROGOSIN, *Mittag-Leffler functions, related topics and applications*, Second Edition, Springer-Verlag GmbH Germany, part of Springer Nature, 2020.

[8] H.J. HAUBOLD, A.M. MATHAI, R.K. SAXENA, *Mittag-Leffler functions and their applications*, J. Appl. Math. (2011), 1–51.

[9] P. HUMBERT, *Quelques résultats relatifs à la fonction de Mittag-Leffler* C. R. Hebd. Seances Acad. Sci. vol. 236, 15 (1953), 1467–1468.

[10] W. KOEPF, *Hypergeometric summation an algorithmic approach to summation and special function identities*, Second Edition, Springer-Verlag London, 2014.

[11] F. MAINARDI, *Fractional calculus and waves in linear viscoelasticity: an introduction to mathematical models*, Copyright 2010 by Imperial College Press.

[12] K.S. MILLER, B. ROSS, *An introduction to the fractional calculus and fractional differential equations*, John Wiley & Sons. Inc., New York, 1993.

[13] G. M. MITTAG-LEFFLER, *Une generalisation de l'integrale de Laplace-Abel*, Comptes Rendus de l'Academie des Sciences Série II, vol. 137, (1903), 537–539.

[14] G. M. MITTAG-LEFFLER, *Sur la nouvelle fonction* $E_\alpha(z)$, Comptes Rendus de l'Academie des Sciences, vol. 137, (1903), 554–558.

[15] Y.L. LUKE, *Special functions and their approximations*, Vol. 1, Ed. Academic Press, 1969.

[16] K. OLDHAM, J. SPANIER, *The fractional calculus, theory and applications of differentiation and integration of arbitrary order*, Academic Press, USA, 1974.

[17] I. PODLUBNY, *The Laplace transform method for linear differential equations of the fractional order*. arXiv: Functional Analysis, 1997.

[18] X.J. YANG, *Theory and applications of special functions for scientists and engineers*, Springer; 1st ed. , 2021.

[19] X.J. YANG, *An introduction to hypergeometric, supertrigonometric and superhyperbolic functions*, Academic Press Elsevier, 2021.

[20] A. WIMAN, *Uber den fundamental satz in der theorie der funcktionen Ea(x)* Acta Math. 29, (1905), 191–201.