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Coupled fixed point theorems on complex partial metric space using different type of contractive conditions

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Abstract: In this paper, we obtain coupled fixed point theorems on complex partial metric space using different type of contractive conditions. An example to support our result is presented.

Keywords: Coupled fixed point; complex partial metric space.

1 Introduction

In many branches of science, economics, computer science, engineering and the development of nonlinear analysis, the fixed point theory is one of the most important tool. In 1989, Backhtin [2] introduced the concept of b-metric space. In 1993, Czerwik [3] extended the results of b-metric spaces. Azam et al.[4] introduced new spaces called complex valued metric spaces and established the existence of fixed point theorems under the contraction condition. P. Dhivya and M. Marudai [5] introduced new spaces called complex partial metric space and established the existence of common fixed point theorems under the contraction condition of rational expression. Bhaskar and Lakshmikantham [7] introduced the concept of coupled fixed point. Ćirić and Lakshmikantham [8] investigated some more coupled fixed point theorems in partially ordered sets. Hassen Aydi [1] introduced coupled fixed point results on partial metric spaces. In this paper, we introduced coupled fixed point results on complex partial metric spaces under the contractive condition.

2 Preliminaries

Let $\mathbb{C}$ be the set of complex numbers and $c_1, c_2 \in \mathbb{C}$. Define a partial order $\preceq$ on $\mathbb{C}$ as follows:

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$c_1 \preceq c_2$ if and only if $Re(c_1) \leq Re(c_2)$ and $Im(c_1) \leq Im(c_2)$.
Consequently,one can infer that $c_1 \preceq c_2$ if one of the following conditions is satisfied:
(i) $Re(c_1) = Re(c_2), Im(c_1) < Im(c_2)$,
(ii)$Re(c_1) < Re(c_2), Im(c_1) = Im(c_2)$,
(iii)$Re(c_1) < Re(c_2), Im(c_1) < Im(c_2)$,
(iv)$Re(c_1) = Re(c_2), Im(c_1) = Im(c_2)$.
In particular,we write $c_1 \precnsim c_2$ if $c_1 \neq c_2$ and one of $(i),(ii)$ and (iii) is satisfied and we write $c_1 \prec c_2$ if only (iii) is satisfied.Notice that
(a) If $0 \preceq c_1 \precnsim c_2$,then $|c_1| < |c_2|$,
(b) If $c_1 \preceq c_2$ and $c_2 \prec c_3$ then $c_1 \prec c_3$,
(c) If $a, b \in \mathbb{R}$ and $a \leq b$ then $ac \preceq bc$ for all $c \in \mathbb{C}$.

Definition 2.1. *[5] A complex partial metric on a non-empty set U is a function $\xi_c : U \times U \to \mathbb{C}^+$ such that for all $p, r, s \in Y$:*
(i) $0 \preceq \xi_c(p,p) \preceq \xi_c(p,r)(smallself-distances)$
(ii) $\xi_c(p,r) = \xi_c(r,p)(symmetry)$
(iii) $\xi_c(p,p) = \xi_c(p,r) = \xi_c(r,r)$ if and only if $p = r(equality)$
(iv) $\xi_c(p,r) \preceq \xi_c(p,s) + \xi_c(s,r) - \xi_c(s,s)(triangularity)$.
A complex partial metric space is a pair (U, ξ_c) such that U is a non empty set and ξ_c is complex partial metric on U.

For the complex partial metric ξ_c on U, the function $d_{\xi_c} : U \times U \to \mathbb{C}^+$ given by $\xi_c^t = 2\xi_c(p,r) - \xi_c(p,p) - \xi_c(r,r)$ is a (usual) metric on U. Each complex partial metric ξ_c on U generates a topology τ_{ξ_c} on U with the base family of open ξ_c-balls $\{B_{\xi_c}(p,\varepsilon) : p \in U, \varepsilon > 0\}$, where $B_{\xi_c}(p,\varepsilon) = \{r \in U : \xi_c(p,r) < \xi_c(p,p) + \varepsilon\}$ for all $p \in U$ and $0 < \varepsilon \in \mathbb{C}^+$.

Definition 2.2. *[5] Let (U, ξ_c) be a complex partial metric space(CPMS). A sequence (p_n) in a CPMS (U, ξ_c) is converges to $p \in U$, if for every $0 \prec \varepsilon \in \mathbb{C}^+$ there is $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ we get $p_n \in B_{\xi_c}(p,\varepsilon)$*

Definition 2.3. *[5] Let (U, ξ_c) be a complex partial metric space. A sequence (p_n) in a CPMS (U, ξ_c) is called Cauchy if there is $a \in \mathbb{C}^+$ such that for every $\varepsilon \prec 0$ there is $N \in \mathbb{N}$ such that for all $n, m \geq N$ $|\xi_c(p_n, p_m) - a| < \varepsilon$.*

Definition 2.4. *[5] Let (U, ξ_c) be a complex partial metric space(CPMS).*
(1) A CPMS (U, ξ_c) is said to be complete if a Cauchy sequence (p_n) in U converges, with respect to τ_{ξ_c}, to a point $p \in U$ such that $\xi_c(p,p) = \lim_{n,m \to \infty} \xi_c(p_n, p_m)$.
(2) A mapping $H : U \to U$ is said to be continuous at $p_0 \in U$ if for every $\varepsilon \prec 0$, there exist $\delta > 0$ such that $H(B_{\xi_c}(p_0, \delta)) \subset B_{\xi_c}(H(p_0, \varepsilon))$.

Lemma 2.1. *[5] Let (U, ξ_c) be a complex partial metric space. A sequence $\{y_n\}$ is Cauchy sequence in the CPMS (U, ξ_c) then $\{y_n\}$ is Cauchy in a metric space (U, ξ_c^t).*

Definition 2.5. *Let (U, ξ_c) be a complex partial metric space(CPMS).Then an element $(p, r) \in U \times U$ is said to be a coupled fixed point of the mapping $F : U \times U \to U$ if $F(p, r) = p$ and $F(r, p) = r$.*

Theorem 2.2. *Let (U, ξ_c) be a complete complex partial metric space. Suppose that the mapping $\phi : U \times U \to U$ satisfies the following contractive condition for all $\alpha, \beta, \gamma, \delta \in U$*

$$\xi_c(\phi(\alpha, \beta), \phi(\gamma, \delta)) \preceq k\xi_c(\phi(\alpha, \beta), \alpha) + l\xi_c(\phi(\gamma, \delta), \gamma),$$

where k, l are nonnegative constants with $k + l < 1$. Then, ϕ has a unique coupled fixed point.

Proof. Choose $u_0, v_0 \in U$ and set $u_1 = \phi(u_0, v_0)$ and $v_1 = \phi(v_0, u_0)$.Continuing this process, set $u_{n+1} = \phi(u_n, v_n)$ and $v_{n+1} = \phi(v_n, u_n)$.
Then,

$$\begin{aligned}
\xi_c(u_n, u_{n+1}) &= \xi_c(\phi(u_{n-1}, v_{n-1}), \phi(u_n, v_n)) \\
&\preceq k\xi_c(\phi(u_{n-1}, v_{n-1}), u_{n-1}) + l\xi_c(\phi(u_n, v_n), u_n) \\
&= k\xi_c(u_n, u_{n-1}) + l\xi_c(u_{n+1}, u_n) \\
\xi_c(u_n, u_{n+1}) &\preceq \frac{k}{1-l}\xi_c(u_n, u_{n-1})
\end{aligned}$$

which implies that

$$|\xi_c(u_n, u_{n+1})| \leq p|\xi_c(u_{n-1}, u_n)| \tag{1}$$

where $p = \frac{k}{1-l} < 1$. Similarly, one can prove that

$$|\xi_c(v_n, v_{n+1})| \leq p|\xi(v_{n-1}, v_n)| \tag{2}$$

From (1) and (2), we get

$$|\xi_c(u_n, u_{n+1})| + |\xi_c(v_n, v_{n+1})| \leq p(|\xi_c(u_{n-1}, u_n)| + |\xi_c(v_{n-1}, v_n))$$

where $p < 1$.
Also,

$$|\xi_c(u_{n+1}, v_{n+2})| \leq p|\xi_c(u_n, u_{n+1})| \tag{3}$$

$$|\xi_c(v_{n+1}, v_{n+2})| \leq p|\xi_c(v_n, v_{n+1})| \tag{4}$$

From (3) and (4), we get

$$|\xi_c(u_{n+1}, v_{n+2})| + |\xi_c(v_{n+1}, v_{n+2})| \leq p(|\xi_c(u_n, u_{n+1})| + |\xi_c(v_n, v_{n+1})|)$$

Repeating this way, we get

$$\begin{aligned}|\xi_c(u_n,v_{n+1})|+|\xi_c(v_n,v_{n+1})| &\leq p(|\xi_c(v_{n-1},v_n)|+|\xi_c(u_{n-1},u_n)|)\\ &\leq p^2(|\xi_c(v_{n-2},v_{n-1})|+|\xi_c(u_{n-2},u_{n-1})|)\\ &\leq \cdots \leq p^n(|\xi_c(v_0,v_1)|+|\xi_c(u_0,u_1)|)\end{aligned}$$

Now, if $|\xi_c(u_n,v_{n+1})|+|\xi_c(v_n,v_{n+1})| = t_n$, then

$$t_n \leq pt_{n-1} \leq \cdots \leq p^n t_0 \tag{5}$$

If $t_0 = 0$ then $|\xi_c(u_0,u_1)|+|\xi_c(v_0,v_1)| = 0$. Hence $u_0 = u_1 = \phi(u_0,v_0)$ and $v_0 = v_1 = \phi(v_0,v_0)$, which implies that (u_0,v_0) is a coupled fixed point of ϕ.
Let $t_0 > 0$. For each $n \geq m$,we have

$$\begin{aligned}\xi_c(u_n,u_m) &\preceq \xi_c(u_n,u_{n-1})+\xi_c(u_{n-1},u_{n-2})-\xi_c(u_{n-1},u_{n-1})\\ &+\xi_c(u_{n-2},u_{n-3})+\xi_c(u_{n-3},u_{n-4})-\xi_c(u_{n-3},u_{n-3})\\ &+\cdots+\xi_c(u_{m+2},u_{m+1})+\xi_c(u_{m+1},u_m)-\xi_c(u_{m+1},u_{m+1})\\ &\preceq \xi_c(u_n,u_{n-1})+\xi_c(u_{n-1},u_{n-2})+\cdots+\xi_c(u_{m+1},u_m)\end{aligned}$$

which implies that

$$|\xi_c(u_n,u_m)| \leq |\xi_c(u_n,u_{n-1})|+|\xi_c(u_{n-1},u_{n-2})|+\cdots+|\xi_c(u_{m+1},u_m)|.$$

Similarly,one can prove that

$$|\xi_c(v_n,v_m)| \leq |\xi_c(v_n,v_{n-1})|+|\xi_c(v_{n-1},v_{n-2})|+\cdots+|\xi_c(v_{m+1},v_m)|.$$

Thus,

$$\begin{aligned}|\xi_c(u_n,u_m)|+|\xi_c(v_n,v_m)| &\leq t_{n-1}+t_{n-2}+t_{n-3}+\cdots+t_m\\ &\leq (p^{n-1}+p^{n-2}+\cdots+p^m)t_0\\ &\leq \frac{p^m}{1-p}t_0 \to 0 \quad n \to \infty.\end{aligned}$$

which implies that $\{u_n\}$ and $\{v_n\}$ are Cauchy sequence in (U,ξ_c). Since the partial metric space (U,ξ_c) is complete, there exists $u,v \in U$ such that $\{u_n\} \to u$ and $v_n \to v$ as $n \to \infty$ and $\xi_c(u,u) = \lim_{n\to\infty}\xi_c(u,u_n) = \lim_{n,m\to\infty}\xi_c(u_n,u_m) = 0$,$\xi_c(u,u) = \lim_{n\to\infty}\xi_c(v,v_n) = \lim_{n,m\to\infty}\xi_c(v_n,v_m) = 0$. We now show that $u = \phi(p,q)$. We suppose on the contrary that

$u \neq \phi(u,v)$ and $v \neq \phi(v,u)$ so that $0 \prec \xi_c(u,\phi(u,v)) = l_1$ and $0 \prec \xi_c(v,\phi(v,u)) = l_2$ then

$$\begin{aligned} l_1 = \xi_c(u,\phi(u,v)) &\preceq \xi_c(u,u_{n+1}) + \xi_c(u_{n+1},\phi(u,v)) \\ &= \xi_c(u,u_{n+1}) + \xi_c(\phi(u_n,v_n),\phi(u,v)) \\ &\preceq \xi_c(u,u_{n+1}) + k\xi_c(u_{n-1},u_n) + l\xi_c(\phi(u,v),u) \\ &\preceq \frac{1}{1-l}\xi_c(u,u_{n+1}) + \frac{k}{1-l}\xi(u_{n-1},u_n) \end{aligned}$$

which implies that

$$|l_1| \leq \frac{1}{1-l}|\xi_c(u,u_{n+1})| + \frac{k}{1-l}|\xi(u_{n-1},u_n)|$$

As $n \to \infty$, $|l_1| \leq 0$. Which is a contradiction, therefore $|\xi_c(u,\phi(u,v))| = 0$ implies $u = \phi(u,v)$. Similarly we can prove that $v = \phi(v,u)$. Thus (u,v) is a coupled fixed point of ϕ. Now, if (g,h) is another coupled fixed point of ϕ,then

$$\begin{aligned} \xi_c(u,g) = \xi_c(\phi(u,v),\phi(g,h)) &\preceq k\xi_c(\phi(u,v),u) + l\xi_c(\phi(g,h),g) \\ &= k\xi_c(u,u) + l\xi_c(g,g) = 0 \end{aligned}$$

Thus,we have $g = u$. Similarly, we get $h = v$. Therefore ϕ has a unique coupled fixed point . □

Corollary 2.3. *Let (U,ξ_c) be a complete complex partial metric space. Suppose that the mapping $\phi : U \times U \to U$ satisfies the following contractive condition for all $\alpha,\beta,\gamma,\delta \in U$*

$$\xi_c(\phi(\alpha,\beta),\phi(\gamma,\delta)) \preceq \frac{k}{2}(\xi_c(\phi(\alpha,\beta),\alpha) + \xi_c(\phi(\gamma,\delta),\gamma)), \tag{6}$$

where $0 \leq k < 1$. Then, ϕ has a unique coupled fixed point.

Example 2.4. *Let $U = [0,\infty)$ endowed with the usual complex partial metric $\xi_c \colon U \times U \to [0,\infty)$ defined by $\xi_c(p,q) = \max\{p,q\}(1+i)$.The complex partial metric space (U,ξ_c) is complete because (U,ξ_c^t) is complete. Indeed,for any $p,q \in U$,*

$$\begin{aligned} \xi_c^t &= 2\xi_c(p,r) - \xi_c(p,p) - \xi_c(r,r) \\ &= 2\max\{p,q\}(1+i) - (p+ip) - (q+iq) \\ &= |p-q| + i|p-q|. \end{aligned}$$

Thus, (U,ξ_c) is the Euclidean complex metric space which is complete. Consider the mapping $\phi\colon U\times U\to U$ defined by $\phi(p,q)=\frac{p+q}{12}$. For any $p,q,g,h\in U$, we have

$$\begin{aligned}\xi_c(\phi(p,q),\phi(g,h)) &= \frac{1}{12}\max\{p+g,\phi(p,q)+\phi(g,h)\}(1+i)\\ &\le \frac{1}{12}[\max\{\phi(p,q),p\}+\max\{\phi(g,h),g\}](1+i)\\ &= \frac{1}{12}[\xi_c(\phi(p,q),p)+\xi_c(\phi(g,h),g)].\end{aligned}$$

which is the contractive condition (6) *for $k=\frac{1}{6}$. Therefore, by Corollary 2.3, and hence ψ has a unique coupled fixed point, which is $(0,0)$. Note that if the mapping $\phi\colon U\times U\to U$ is given by $\phi(p,q)=\frac{p+q}{2}$, then ϕ satisfies the contractive condition* (6) *for $k=1$, that is,*

$$\begin{aligned}\xi_c(\phi(p,q),\phi(g,h)) &= \frac{1}{2}\max\{p+g,\phi(p,q)+\phi(g,h)\}(1+i)\\ &\le [\max\{\phi(p,q),p\}+\max\{\phi(g,h),g\}](1+i)\\ &= \frac{1}{2}[\xi_c(\phi(p,q),p)+\xi_c(\phi(g,h),g)].\end{aligned}$$

In this case, $(0,0)$ and $(1,1)$ are both coupled fixed points of ϕ, and, hence, the coupled fixed point of ϕ is not unique. This shows that the condition $k<1$ in Corollary 2.3, and hence $k+l<1$ in Theorem 2.2 cannot be omitted in the statement of the aforesaid results.

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