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Selective Properties of Fuzzy 2-Metric Spaces

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Abstract: We introduce and study some selective covering properties in fuzzy 2-metric spaces. These properties are related to the classical covering properties of Menger, Hurewicz and Rothberger which are well known in selection principles theory.

Keywords: F-2-Menger bounded, F-2-Hurewicz bounded, F-2-Rothberger bounded, game theory

1 Introduction

In this paper we study some topological properties of fuzzy 2-metric spaces related to the classical covering properties of Menger, Hurewicz and Rothberger (for these properties see the survey articles [7, 11]). Recall that a topological space has the *Menger* (resp., *Hurewicz*) *covering property* if for each sequence $(\mathscr{U}_n)_{n\in\mathbb{N}}$ of open covers of X there is a sequence $(\mathscr{V}_n)_{n\in\mathbb{N}}$ such that for each n, $\mathscr{V}_n$ is a finite subset of $\mathscr{U}_n$ and $X=\bigcup_{n\in\mathbb{N}}\bigcup\mathscr{V}_n$ (resp., each $x\in X$ belongs to $\cup\mathscr{V}_n$ for all but finitely many n). X has the *Rothberger property* if for each sequence $(\mathscr{U}_n)_{n\in\mathbb{N}}$ of open covers of X there are $U_n\in\mathscr{U}_n$, $n\in\mathbb{N}$, such that $X=\bigcup_{n\in\mathbb{N}}U_n$.

In the 1960s, Gähler introduced the notion of 2-metric space [5, 6].

Let X be a non-empty set and let $d: X\times X\times X\to\mathbb{R}$ be a mapping satisfying the following conditions:

1. For every pair of distinct points $x,y\in X$ there exists a point $z\in X$ such that $d(x,y,z)\neq 0$;

2. $d(x,y,z)=0$ only if at least two of three points are the same;

3. $d(x,y,z)=d(x,z,y)=d(y,x,z)=d(y,z,x)=d(z,x,y)=d(z,y,x)$ for all $x,y,z\in X$ (the symmetry);

4. $d(x,y,z)\leq d(w,y,z)+d(x,w,z)+d(x,y,w)$ for all $x,y,z,w\in X$ (the tetrahedral inequality).

Then d is called a 2-*metric* on X and (X,d) is called a 2-*metric space*.

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In this paper, we assume that all 2-metric spaces have at least three distinct points. Observe also that every 2-metric is non-negative.

A typical example of 2-metric spaces is the Euclidean plane $\mathbb{R}^2$ with the 2-metric d defined as the area of the triangle spanned by points $x,z,y \in \mathbb{R}^2$. Another such example is $\mathbb{R}^3$ with $d(x,y,z) = \min\{|x-y|,|y-z|,|z-x|\}$. Nowadays there is the large literature on 2-metric spaces and their modifications, mainly in fixed point theory (see, for example, [1, 2, 3, 4, 9, 9, 10, 12]).

To define fuzzy 2-metric spaces we need the well-known notion of triangular or t-norm.

Definition 1.1 ([14]) A binary operation $* : [0,1]\times[0,1] \to [0,1]$ is a *continuous t-norm* if the following conditions are satisfied:

(1) $*$ is commutative and associative;

(2) $*$ is continuous;

(3) $a*1 = a$ for all $a \in [0,1]$;

(4) $a*b \le c*d$ whenever $a \le c$ and $b \le d$, $(a,b,c,d \in [0,1])$.

Definition 1.2 ([13]) A 3-tuple $(X,M,*)$ is said to be a *fuzzy* 2*-metric space* if X is an arbitrary nonempty set, $*$ is a continuous t-norm, and M is a fuzzy set on $X^3 \times (0,\infty)$ satisfying $(x,y,z \in X,\ t,t_1,t_2,t_3 \in (0,\infty))$ the following conditions:

(F2M.1) given distinct elements $x,y \in X$ there is an element $z \in X$ such that $M(x,y,y,t) > 0$ for each $t > 0$;

(F2M.2) $M(x,y,z,t) = 1$ if at least two of x,y,z are equal;

(F2M.3) $M(x,y,z,t) = M(x,z,y,t) = M(y,z,x,t)$ for all $x,y,z \in X$ and all $t > 0$;

(F2M.4) $M(x,y,z,t_1+t_2+t_3) \ge M(x,z,w,t_1) * M(x,w,z,t_2) * M(w,y,z,t_3)$;

(F2M.5) $M(x,y,z,\cdot) : (0,\infty) \to (0,1]$ is a continuous function.

The pair $(M,*)$ (or only M) is called a 2*-fuzzy metric* on X.

The following is a typical example of a fuzzy 2-metric.

Example 1.3 Let (X,d) be a 2-metric space. Then the mapping $M_d : X^3 \times (0,\infty) \to [0,1]$ defined by

$$M_d(x,y,z,t) = \frac{t}{t+d(x,y,z)},\ (x,y,z \in X, t > 0)$$

is a fuzzy 2-metric on X induced by the 2-metric d.

Let $(X,M,*)$ be a fuzzy 2-metric space, $x \in X$, $S \subset X$, $r \in (0,1)$, $t > 0$. The set

$$B(x,r,t) = \{y \in X : M(x,y,z,t) > 1-r \text{ for each } z \in X\}$$

is called the *open ball* with center x and radius r with respect to t.

The collection of all open balls with center x, $x \in X$, is a base for a topology on $(X,M,*)$, denoted by τ_M.

We also define

$$B(S,\varepsilon,t) := \bigcup_{x\in S} B(x,\varepsilon,t).$$

2 Results

In this section we define and study some covering properties of fuzzy 2-matric spaces.

Definition 2.1 A fuzzy 2-metric space $(X, M, *)$ is said to be:

FM_2 : F-2-*Menger-bounded* (or FM_2-bounded);
FR_2 : F-2-*Rothberger-bounded* (or FR_2-bounded);
FH_2 : F-2-*Hurewicz-bounded* (or FH_2-bounded)

if for each sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of elements of $(0,1)$ and each $t > 0$ there is a sequence

FM_2 : $(A_n)_{n\in\mathbb{N}}$ of finite subsets of X such that $X = \bigcup_{n\in\mathbb{N}}\bigcup_{a\in A_n} B(a,\varepsilon_n,t)$;
FR_2 : $(x_n)_{n\in\mathbb{N}}$ of elements of X such that $X = \bigcup_{n\in\mathbb{N}} B(x_n,\varepsilon_n,t)$;
FH_2 : $(A_n)_{n\in\mathbb{N}}$ of finite subsets of X such that for each $x \in X$ there is $n_0 \in \mathbb{N}$ such that $x \in \bigcup_{a\in A_n} B(a,\varepsilon_n,t)$ for all $n \geq n_0$.

Recall that a fuzzy 2-metric space is said to be *fuzzy* 2*-precompact* (respectively, *fuzzy* 2*-pre-Lindelöf*) if for every $\varepsilon \in (0,1)$ and every $t > 0$ there is a finite (respectively, countable) set $A \subset X$ such that $X = \bigcup_{a\in A} B(a,\varepsilon,t)$.

Evidently,

$$\mathsf{F}-2\text{-precompact} \Rightarrow \mathsf{FH}_2-\text{bounded} \Rightarrow \mathsf{FM}_2-\text{bounded} \Rightarrow \mathsf{F}-2\text{-pre-Lindelöf}$$

and

$$\mathsf{FR}_2-\text{bounded} \Rightarrow \mathsf{FM}_2-\text{bounded}.$$

Example 2.2 Let (X,d) be a 2-metric space with the Menger property (with respect to the topology τ_d). Then the induced fuzzy 2-metric space $(X, M_d, *)$ with $* = \cdot$ (the product t-norm) is FM_2-bounded.

Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers and $t > 0$. Applied the fact that (X,d) has the Menger covering property to the sequence $(\mathscr{U}_n)_{n\in\mathbb{N}}$, $\mathscr{U}_n = \{B(x,\varepsilon_n) : x \in X\}$, to find a sequence $(A_n)_{n\in\mathbb{N}}$ of finite subsets of X such that

$$X = \bigcup_{n\in\mathbb{N}}\bigcup_{a\in A_n} B(a,\varepsilon_n).$$

Let $x \in X$. There is $k \in \mathbb{N}$ and a point $a_k \in A_k$ such that $x \in B(a_k,\varepsilon_k)$, i.e. $\sup_{z\in X} d(a_k,x,z) < \varepsilon_k$. Then for all $z \in X$ we have

$$M_d(x,a_k,z,t) = \frac{t}{t+d(x,a_k,z)} > \frac{t}{t+\varepsilon_k} = 1 - \frac{\varepsilon_k}{t+\varepsilon_k} > 1-\varepsilon_k.$$

Therefore, $x \in B(a_k,\varepsilon_k,t)$, i.e. $X = \bigcup_{n\in\mathbb{N}}\bigcup_{a\in A_n} B(a,\varepsilon_n,t)$. This means that $(X, M_d, *)$ is FM_2-bounded.

Remark 2.3 Similarly we prove: If a 2-metric space (X,d) has the Hurewicz (Rothberger) property, then the induced fuzzy 2-metric space $(X,M_d,*)$ with $* = \cdot$ is FH_2-bounded (FR_2-bounded).

Theorem 2.4 *For a fuzzy 2-metric space $(X,M,*)$ the following are equivalent:*

(a) *For each sequence $(\varepsilon_n)_{n\in\mathbb{N}} \subset (0,1)$ and each $t>0$ there is a sequence $(A_n)_{n\in\mathbb{N}}$ of finite subsets of X such that each finite subset $F \subset X$ is contained in $B(A_n,\varepsilon_n,t)$ for some A_n;*

(b) *For each sequence $(\varepsilon_n)_{n\in\mathbb{N}} \subset (0,1)$ and each $t>0$ there is a sequence $(A_n)_{n\in\mathbb{N}}$ of finite subsets of X and an increasing sequence $n_1 < n_2 < \cdots$ of natural numbers such that each finite subset $F \subset X$ is contained in $\bigcup_{n_k \le i < n_{k+1}} B(A_i,\varepsilon_i,t)$ for some $k \in \mathbb{N}$.*

Proof. Evidently (a) implies (b). We prove $(b) \Rightarrow (a)$. Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence of elements from $(0,1)$ and $t>0$. For each $n\in\mathbb{N}$ let $\mu_n = \min\{\varepsilon_i : i \le n\}$ and apply (b) to $(\mu_n)_{n\in\mathbb{N}}$ and t. There is an increasing sequence $n_1 < n_2 < \cdots$ in $\mathbb{N}$ such that each finite set $F \subset X$ is contained in $\bigcup_{n_k \le i < n_{k+1}} B(A_i,\delta_i,t)$ for some $k\in\mathbb{N}$. Define now

$C_n = \bigcup_{i<n_1} A_i$, for each $n < n_1$,

$C_n = \bigcup_{n_k \le i < n_{k+1}} A_i$, for each n such that $n_k \le n < n_{k+1}$.

We claim that the sequence $(C_n)_{n\in\mathbb{N}}$ of finite subsets of X witnesses for $(\varepsilon_n)_{n\in\mathbb{N}}$ and t that (a) is satisfied.

Let F be a finite subset of X. Choose $k\in\mathbb{N}$ such that $F \subset \bigcup_{n_k \le i < n_{k+1}} B(A_i,\mu_i,t)$. For (each) n with $n_k \le n < n_{k+1}$ put $C_n = \bigcup_{n_k \le i < n_{k+1}} S_i$. We have that for each $x\in F$ there is j, $n_k \le j < n_{k+1}$, and $y \in A_j$ with $x \in B(y,\mu_j,t)$. Further, we have $B(y,\mu_j,t) \subset B(y,\varepsilon_j,t)$ and since $y \in C_n$ we have $x \in B(C_j,\varepsilon_j,t)$, and thus $F \subset B(C_j,\varepsilon_j,t)$. □

With a small modification in the previous proof one can prove the following.

Theorem 2.5 *For a fuzzy 2-metric space $(X,M,*)$ the following are equivalent:*

(a) *For each sequence $(\varepsilon_n)_{n\in\mathbb{N}} \subset (0,1)$ and each $t>0$ there is a sequence $(A_n)_{n\in\mathbb{N}}$ of finite subsets of X such that each finite subset $F \subset X$ is contained in $B(A_n,\varepsilon_n,t)$ for all but finitely many n;*

(b) *For each sequence $(\varepsilon_n)_{n\in\mathbb{N}} \subset (0,1)$ and each $t>0$ there is a sequence $(A_n)_{n\in\mathbb{N}}$ of finite subsets of X and an increasing sequence $n_1 < n_2 < \cdots$ of natural numbers such that each finite subset $F \subset X$ is contained in $\bigcup_{n_k \le i < n_{k+1}} B(A_i,\varepsilon_i,t)$ for all but finitely many $k\in\mathbb{N}$.*

Definition 2.6 Let $(X,M,*)$ be a fuzzy 2-metric space and $Y \subset X$. Then the mapping $M_Y = M \restriction Y^3 \times (0,\infty) : Y^3 \times (0,\infty) \to [0,1]$ satisfying (F2M.1) is also a fuzzy 2-metric on Y, and $(Y,M_Y,*)$ is called the *fuzzy* 2-*metric subspace* of $(X,M,*)$.

Theorem 2.7 *Every fuzzy* 2*-metric subspace of an* FM_2*-bounded fuzzy* 2*-metric space* $(X,M,*)$ *is also* FM_2*-bounded.*

Proof. Let $(Y, M_Y, *)$ be a fuzzy 2-metric subspace of $(X, M, *)$ and let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence of elements of $(0,1)$ and $t > 0$. Since the t-norm $*$ is continuous, for each $n \in \mathbb{N}$ there is $\eta_n \in (0,1)$ such that $(1-\eta_n) * (1-\eta_n) * (1-\eta_n) > 1-\varepsilon_n$. By assumption on $(X, M, *)$ (applied to the sequence $(\eta_n)_{n\in\mathbb{N}}$ and $\frac{t}{3}$) there is a sequence $(A_n)_{n\in\mathbb{N}}$ of finite subsets of X such that

$$X = \bigcup_{n\in\mathbb{N}} \bigcup_{a\in A_n} B(a, \eta_n, t/3).$$

For each $n \in \mathbb{N}$ let

$$C_n = \{c \in A_n : \exists y \in Y \text{ with } y \in B(c, \eta_n, t/3)\}.$$

Further, for each $c \in C_n$ pick an element $y_c \in Y$ such that $y_c \in B(c, \eta_n, t/3)$ and set

$$D_n = \{y_c : c \in C_n\}.$$

Let us show that the sequence $(D_n)_{n\in\mathbb{N}}$ of finite subsets of Y witnesses for $(\varepsilon_n)_{n\in\mathbb{N}}$ and $t > 0$ that $(Y, M_Y, *)$ is FM_2-bounded.

Let y be an arbitrary element of Y. There exist $n \in \mathbb{N}$ and $a \in A_n$ such that $y \in B(a, \eta_n, t/3)$, and from the definition of C_n it follows $a \in C_n$. Therefore, there exists $y_a \in D_n$ such that $y_a \in B(a, \eta_n, t/3)$, hence $a \in B(y_a, \eta_n, t/3)$. So, we have

$$M(a, y, z, t/3) > 1-\eta_n \text{ and } M(a, y_a, z, t/3) > 1-\eta_n.$$

Applying the tetrahedral inequality (F2M.4), we have

$$M(y, y_a, z, t) \geq M(y, y_a, a, t/3) * M(y, a, z, t/3) * M(a, y_a, z, t/3)$$

$$> (1-s_n) * (1-s_n) * (1-s_n) > 1-\varepsilon_n,$$

which means $y \in B(y_a, \varepsilon_n, t)$. As $y \in Y$ was arbitrary we conclude

$$Y = \bigcup_{n\in\mathbb{N}} \bigcup_{y_a\in D_n} B(y_a, \varepsilon_n, t),$$

i.e. $(Y, M_Y, *)$ is FM_2-bounded. □

The proof of the following theorem is similar to the proof of Theorem 2.7 and thus it is omitted.

Theorem 2.8 *Every fuzzy 2-metric subspace of an* FH_2*-bounded space* $(X, M, *)$ *is also* FH_2*-bounded.*

Let $(X, M_X, *)$ and $(Y, M_Y, *)$ be fuzzy 2-metric spaces and let $Z = X \times Y$. Then the mapping $M_Z : Z^3 \times (0, \infty) \to [0,1]$ defined by

$$M_Z(z_1, z_2, z_3, t) = M_X(x_1, x_2, x_3, t) * M_Y(y_1, y_2, y_3, t)$$

for all $z_i = (x_i, y_i) \in Z$, $i = 1,2,3$, and all $t > 0$ is a fuzzy 2-metric on Z, and the triple $(Z, M_Z, *)$ is called the *product 2-metric space* of X and Y.

Theorem 2.9 *The product* $(Z, M_Z, *)$ *of two* FH_2*-bounded spaces* $(X, M_X, *)$ *and* $(Y, M_Y, *)$ *is also* FH_2*-bounded.*

Proof. Let a sequence $(\varepsilon_n)_{n\in\mathbb{N}} \subset (0,1)$ and $t > 0$ be given. By continuity of $*$, choose for each $n \in \mathbb{N}$ an element η_n in $(0,1)$ such that $(1-\eta_n)*(1-\eta_n) > 1-\varepsilon_n$. By assumption on X and Y there are sequences $(F_n)_{n\in\mathbb{N}}$ and $(H_n)_{n\in\mathbb{N}}$ of finite sets of X and Y, respectively and natural numbers n_1 and n_2 such that each $x \in X$ belongs to $\bigcup_{a\in F_n} B(a, \eta_n, t/2)$ for all $n \geq n_1$, and each $y \in Y$ belongs to $\bigcup_{c\in H_n} B(c, \eta_n, t/2)$ for all $n \geq n_2$. We claim that the sequence $(F_n \times H_n)_{n\in\mathbb{N}}$ of finite subsets of Z witnesses for $(\varepsilon_n)_{n\in\mathbb{N}}$ and t that $(Z, M_Z, *)$ is FH_2-bounded.

Let $z = (x,y) \in Z$ and $n_0 = \max\{n_1, n_2\}$. Then for each $n \geq n_0$

$$x \in B(a_n, \eta_n, t/2) \text{ for some } a_n \in F_n$$

and

$$y \in B(c_n, \eta_n, t/2) \text{ for some } c_n \in H_n.$$

Therefore, for all $n \geq n_0$ and $z_n = (a_n, c_n) \in F_n \times H_n$ we have

$$M_Z(z, z_n, w, t) = M_X(x, a_n, w, t) * M_Y(y, c_n, w, t) > (1-\eta_n)*(1-\eta_n) > 1-\varepsilon_n.$$

This means that $z \in B(z_n, \varepsilon_n, t)$ and one concludes that $(Z, M_Z, *)$ is FH_2-bounded. □

In a similar way, with small necessary changes, one can prove the following.

Theorem 2.10 *The product* $(Z, M_Z, *)$ *of an* FM_2*-bounded fuzzy 2-metric space* $(X, M_X, *)$ *and a fuzzy* 2*-precompact fuzzy 2-metric space* $(Y, M_Y, *)$ *is* FM_2*-bounded.*

We end the paper by two open questions.

There are infinitely long two-person games associated to FM_2-boundedness, FH_2-boundedness, FR_2-boundedness. We describe the game associated to the FR_2-boundedness; it is clear how to define games related to the other two properties.

The game $\mathsf{G}_{\mathsf{FR}_2}$ on a fuzzy 2-metric space $(X, M, *)$ is defined in the following way. Let $t > 0$ be fixed. Two players, I and II, play a round for each positive integer n. In the n-th round I takes $\varepsilon_n \in (0,1)$, and II responds by choosing an element $a_n \in X$. A play $\varepsilon_1, a_1; \varepsilon_2, a_2, \cdots; \varepsilon_n, a_n; \cdots$ is won by II if and only if $X = \bigcup_{n\in\mathbb{N}} B(a_n, \varepsilon_n, t)$.

Evidently, if the player II has a winning strategy (or weaker, if I does not have a winning strategy) in the game $\mathsf{G}_{\mathsf{FR}_2}$, then $(X, M, *)$ is FR_2-bounded.

Call a fuzzy 2-metric space X *strongly* FR_2*-bounded* if II has a winning strategy in the game $\mathsf{G}_{\mathsf{FR}_2}$. Similarly we define *strong* FM_2*-boundedness* and *strong* FH_2*-boundedness*.

Problem 2.11 *Find fuzzy 2-metric spaces which are* FR_2*-bounded* (*respectively,* FM_2*-bounded,* FH_2*-bounded*)*, but not strongly* FR_2*-bounded* (*respectively, strongly* FM_2*-bounded, strongly* FH_2*-bounded*)*.*

Problem 2.12 *Characterize strongly* FR_2*-bounded, strongly* FM_2*-bounded and strongly* FH_2*-bounded fuzzy 2-metric spaces.*

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