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A Survey on Randić (Normalized) Incidence Energy of Graphs

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Abstract: For a graph G of order n with normalized signless Laplacian eigenvalues $\gamma_1^+ \geq \gamma_2^+ \geq \cdots \geq \gamma_n^+ \geq 0$, the Randić (normalized) incidence energy is defined as $I_RE(G) = \sum_{i=1}^{n} \sqrt{\gamma_i^+}$. In this paper, we present a survey on the results of $I_RE(G)$, especially with emphasis on the properties, bounds and Coulson integral formula of $I_RE(G)$.

1 Introduction

All graphs considered in this paper are simple finite undirected graphs. The terminology and notation not defined here can be found in [11].

Let $G = (V,E)$ be a graph with n vertices and m edges. The vertex set and edge set of G are, respectively, denoted by $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{e_1, e_2, \ldots, e_m\}$. Let d_i be the degree of the vertex $v_i \in V$, $i = 1, 2, \ldots, n$. Denote by δ and Δ the minimum degree and the maximum degree of G, respectively. If v_i and v_j are two adjacent vertices of G, then it is written as $i \sim j$. Let $A(G) = (a_{ij})$ be the $(0,1)$-adjacency matrix of the graph G. It is defined by $a_{ij} = 1$ if $i \sim j$ and 0 otherwise. The eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ of $A(G)$ are the (ordinary) eigenvalues of G [11].

For a graph G, the (ordinary) graph energy was introduced as the sum of absolute values of its eigenvalues [15]. It is defined as

$$E(G) = \sum_{i=1}^{n} |\lambda_i|. \tag{1.1}$$

For details on the theory of $E(G)$, see [21] and the references cited therein.

As the generalization of the graph energy concept, the energy of a real matrix (not necessarily square) M, denoted by $E(M)$, is defined by Nikiforov [29] as the sum of its singular values that are equal to the square roots of the eigenvalues of MM^T, where M^T is the transpose of M. Especially, for a graph G, $E(G) = E(A(G))$.

Denote by $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$ the Laplacian matrix and the signless Laplacian matrix of G, respectively [26]. Here, $D(G) = diag(d_1, d_2, \ldots, d_n)$

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is the diagonal degree matrix of G. For a graph G without isolated vertices, the matrix $D(G)^{-1/2}$ is well defined. Then, the normalized Laplacian matrix is defined as

$$\mathscr{L}(G) = D(G)^{-1/2} L(G) D(G)^{-1/2} = I_n - R(G)$$

and the normalized signless Laplacian matrix as [9]

$$\mathscr{L}^{+}(G) = D(G)^{-1/2} Q(G) D(G)^{-1/2} = I_n + R(G)$$

where I_n is the $n \times n$ unit matrix and $R(G)$ is the Randić matrix. Throughout this paper, the eigenvalues of $R(G)$, $\mathscr{L}(G)$ and $\mathscr{L}^{+}(G)$ (or Randić, normalized Laplacian and normalized signless Laplacian eigenvalues of G) will be denoted by $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$, $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_{n-1} > \gamma_n = 0$ and $\gamma_1^{+} \geq \gamma_2^{+} \geq \cdots \geq \gamma_n^{+} \geq 0$, respectively. Details on these eigenvalues can be found in [2, 9, 11].

Let $I(G)$ be the vertex-edge incidence matrix of the graph G. The ij-entry of $I(G)$ is 1 if v_i is incident to e_j and 0 otherwise. The incidence energy of G, denoted by $IE(G)$, is defined as the energy of its incidence matrix [18]. Since $Q(G) = I(G) I(G)^T$, Gutman et al. also discovered that [17]

$$IE(G) = \sum_{i=1}^{n} \sqrt{q_i}\,,$$

where $q_1 \geq q_2 \geq \cdots \geq q_n \geq 0$ are the eigenvalues of $Q(G)$ [12]. For the basic properties and several lower and upper bounds of $IE(G)$, see [4, 13, 18, 23].

Gu et al. [14] and Cheng and Liu [8] independently introduced the Randić (normalized) incidence matrix of G as $I_R(G) = D(G)^{-1/2} I(G)$ and referred to its energy as the Randić (normalized) incidence energy $I_R E(G)$ of G. Since $\mathscr{L}^{+}(G) = I_R(G) I_R(G)^T$, in full analogous manner with the incidence energy, it was also pointed out that [8, 14]

$$I_R E(G) = \sum_{i=1}^{n} \sqrt{\gamma_i^{+}}\,. \tag{1.2}$$

For the recent results on $I_R E(G)$, see [8, 14, 22, 30].

This survey is organized in the following way. In Section 2, we recall some known results regarding the normalized signless Laplacian eigenvalues. In Section 3, we deal with a few elementary properties of $I_R E(G)$. In Sections 4 and 5, some lower and upper bounds for $I_R E(G)$ are given. In Section 6, the results on the Coulson integral formula of $I_R E(G)$ are presented.

2 Some Known Results

In this section, we recall some known results associated with the normalized signless Laplacian eigenvalues of graphs.

Lemma 2.1. [14] *Let G be a graph of order n with no isolated vertices. Then $\gamma_i^{+} = 1 + \rho_i$, for $i = 1, 2, \ldots, n$.*

Lemma 2.2. [6] *If G is a connected non–bipartite graph of order n, then* $\gamma_i^+ > 0$, *for* $i = 1,2,\dots,n$.

Lemma 2.3. [14] *For a graph G with no isolated vertices, the largest normalized signless Laplacian eigenvalue* $\gamma_1^+ = 2$.

Let $\widetilde{G}$ denote the subdivision graph of a graph G obtained by inserting additional vertex into the each edge of G. If G is a graph with n vertices and m edges, then its subdivision graph $\widetilde{G}$ possesses $n+m$ vertices and $2m$ edges.

Lemma 2.4. [8] *Let G be a graph with n vertices and m edges and let* $\widetilde{G}$ *be its subdivision graph. If* γ_i^+ *are the non-zero normalized signless Laplacian eigenvalues of G, then the Randić eigenvalues of* $\widetilde{G}$ *consist of the number* $\pm\sqrt{\gamma_i^+/2}$ $i = 1,2,\dots,h$, *and* $n+m-2h$ *zeros.*

Lemma 2.5. [8] *Let G be a connected graph with diameter d and s distinct normalized signless Laplacian eigenvalues. Then,* $d \le s-1$.

Lemma 2.6. [14] *Let G be a graph of order* $n \ge 2$ *with no isolated vertices. Then* $\gamma_2^+ = \gamma_3^+ = \cdots = \gamma_n^+ = \frac{n-2}{n-1}$ *if and only if* $G \cong K_n$.

Lemma 2.7. [8] *Suppose that the n-vertex connected graph G is not a complete graph. If the normalized signless Laplacian eigenvalues are ordered as* $\gamma_1^+ \ge \gamma_2^+ \ge \cdots \ge \gamma_n^+$, *then* $\gamma_2^+ \ge 1$.

Recall that the general Randić index of a graph G is defined as $R_{-1}(G) = \sum_{i\sim j} \frac{1}{d_i d_j}$ [7]. The following lower bound on the second largest normalized signless Laplacian eigenvalue involving the parameter n and $R_{-1}(G)$ can be found in [24].

Lemma 2.8. [24] *Let G be a connected non–bipartite graph with* $n \ge 3$ *vertices. Then*

$$\gamma_2^+ \ge \frac{n+2R_{-1}(G)-4}{n-2}\,.$$

Equality holds if and only if $G \cong K_n$.

Lemma 2.9. [6] *Let G be a connected non–bipartite graph with* $n \ge 3$ *vertices. Then*

$$\gamma_n^+ \le \frac{n-2R_{-1}(G)}{n} \le \frac{\Delta-1}{\Delta} \le \frac{n-2}{n-1}$$

with equalities if and only if $G \cong K_n$.

Lemma 2.10. [8] *Let G be a graph of order n with no isolated vertices. Then*

$$\sum_{i=1}^{n} \gamma_i^+ = n \quad \text{and} \quad \sum_{i=1}^{n} (\gamma_i^+)^2 = n+2R_{-1}(G)\,.$$

Lemma 2.11. [5] *If G is a bipartite graph, then the eigenvalues of $\mathscr{L}(G)$ and $\mathscr{L}^+(G)$ coincide.*

Let $t(G)$ be the total number of spanning trees of G. Denote by $G_1 \times G_2$ the Cartesian product of the graphs G_1 and G_2.

Lemma 2.12. [5, 11] *Let G be a connected graph with n vertices, m edges and $t(G)$ spanning trees. If G is bipartite, then*

$$\prod_{i=1}^{n-1} \gamma_i = \prod_{i=1}^{n-1} \gamma_i^+ = \frac{2mt(G)}{\prod_{i=1}^{n} d_i}.$$

If G is non–bipartite, then

$$\prod_{i=1}^{n} \gamma_i^+ = \frac{2t(G \times K_2)}{t(G)\prod_{i=1}^{n} d_i}.$$

3 Elementary Properties of $I_RE(G)$

The empty graph of order n is the graph with n isolated vertices and no edges. The Randić (normalized) incidence energy $I_RE(G)$ has similar basic properties as graph energy.

Theorem 3.1. [8, 16] *Let G be a graph of order n. Then*

(a) $E(G) \geq 0$, $I_RE(G) \geq 0$ with equalities if and only if G is an empty graph.

(b) If the graph G consists of two connected components G_1 and G_2, then $E(G) = E(G_1) + E(G_2)$ and $I_RE(G) = I_RE(G_1) + I_RE(G_2)$.

(c) If one component of the graph G is G_1 and all other components are isolated vertices, then $E(G) = E(G_1)$ and $I_RE(G) = I_RE(G_1)$.

By full analogy with the graph energy given by (1.1), the Randić energy of a graph G was defined as follows [2]

$$RE(G) = \sum_{i=1}^{n} |\rho_i|.$$

Considering the result in Lemma 2.4, Cheng and Liu [8] obtained the following relation between the Randić (normalized) incidence energy of a graph and Randić energy of its subdivision graph.

Theorem 3.2. [8] *Let G be a graph with n vertices and m edges and let $\widetilde{G}$ be its subdivision graph. Then, $I_RE(G) = \frac{\sqrt{2}}{2} RE\left(\widetilde{G}\right)$.*

Let α be a real number. The sum of the αth powers of the non-zero normalized Laplacian eigenvalues of a connected graph G was defined as [3]

$$s_\alpha(G) = \sum_{i=1}^{n-1} \gamma_i^\alpha.$$

This sum generalizes various graph invariants. For more details, see [1, 19]. For $\alpha = 1/2$, $s_{1/2}(G)$ is equal to the Laplacian incidence energy of G, defined by [31] (see also, [27,28])

$$LIE(G) = \sum_{i=1}^{n-1} \sqrt{\gamma_i}.$$

Recently, the the sum of the αth powers of the normalized signless Laplacian eigenvalues of G was put forward as [5]

$$\sigma_\alpha(G) = \sum_{i=1}^{n} \left(\gamma_i^+\right)^\alpha.$$

Note that $\sigma_{1/2}(G) = I_R E(G)$, defined by (1.2). Recall from Lemma 2.11 that the normalized Laplacian and the normalized signless Laplacian eigenvalues coincide. By the fact that, the following result was given in [5].

Theorem 3.3. [5] *If G is a bipartite graph, then $\sigma_\alpha(G)$ coincide with $s_\alpha(G)$. In particular, for bipartite graphs, $I_R E(G) = LIE(G)$.*

4 Lower Bounds for $I_R E(G)$

Let K_n and $K_{p,q}$ $(p+q=n)$ be the complete graph and the complete bipartite graph of order n, respectively. We now give some lower bounds on $I_R E(G)$.

Theorem 4.1. [8,14] *Let G be a graph of order n with no isolated vertices. Then, $I_R E(G) \geq \sqrt{n}$ with equality if and only if $G \cong K_2$.*

Theorem 4.2. [8] *Let G be a graph of order n with no isolated vertices. Then,*

$$I_R E(G) \geq \sqrt{\frac{2n^3}{4n-1+(-1)^n}}.$$

Equality holds if and only if n is even and G is disjoint union of $\frac{n}{2}$ paths of length 1.

Corollary 4.1. [8] *Let G be a graph of order n with no isolated vertices. Then*

$$I_R E(G) \geq \frac{n}{\sqrt{2}}. \tag{4.1}$$

Equality holds if and only if n is even and G is disjoint union of $\frac{n}{2}$ paths of length 1.

For a subset $E' \subseteq E$, the subgraph of G obtained by deleting the edges in E' is denoted by $G - E'$. If E' contains only one edge e, then $G - E'$ is denoted by $G - e$.

Theorem 4.3. [14] *Let G be a graph and E' be a nonempty subset of E. Then*

$$I_R E(G) > I_R E\left(G - E'\right).$$

When the edge subset E' contains exactly one edge, Gu et al. [14] established the following result.

Theorem 4.4. [14] *Let G be a connected graph, $e = v_i v_j$ be an edge of G. Then*

$$I_R E(G) \geq \sqrt{\frac{1}{d_i} + \frac{1}{d_j} + [I_R E\,(G-e)]^2}\,.$$

Equality holds if and only if $G \cong K_2$.

The clique number $\omega = \omega(G)$ of a graph G is equal to the number of vertices in a maximum clique. By the fact that $I_R E(K_n) = \sqrt{2} + \sqrt{(n-1)(n-2)}$, the following result was presented in [14].

Corollary 4.2. [14] *Let G be a non-empty graph with clique number ω. Then,*

$$I_R E(G) \geq \sqrt{2} + \sqrt{(\omega-1)(\omega-2)}.$$

In particular, if G has at least one edge then $I_R E(G) \geq \sqrt{2}$.

The following relation exists between $I_R E(G)$ and Laplacian incidence energy.

Theorem 4.5. [31] *Let G be a connected graph of order n. Then*

$$I_R E(G) \geq LIE\,(G)\,.$$

Equality holds if G is a bipartite graph [5].

Corollary 4.3. [31] *Let G be a connected graph of order n with minimum degree δ and γ_1 the spectral radius of $\mathscr{L}(G)$. Then*

$$I_R E(G) \geq n \max\left\{\gamma_1^{-1/2}, \sqrt{\frac{\delta}{\delta+1}}\right\}. \tag{4.2}$$

Remark 4.1. [31] *The lower bound* (4.2) *is stonger than the lower bound* (4.1)*.*

Theorem 4.6. [24] *Let G be a connected non–bipartite graph with $n \geq 3$ vertices. Then, for any α, $\gamma_2^+ \geq \alpha \geq 1$, holds*

$$I_R E\,(G) > \sqrt{2} + \sqrt{\alpha} + n - 2 + \frac{1}{2}\ln\left(\frac{t(G \times K_2)}{\alpha\, t(G) \prod_{i=1}^{n} d_i}\right).$$

Corollary 4.4. [24] *Let G, $G \ncong K_n$, be a connected non–bipartite graph with $n \geq 3$ vertices. Then,*

$$I_R E\,(G) > \sqrt{2} + n - 1 + \frac{1}{2}\ln\left(\frac{t(G \times K_2)}{t(G) \prod_{i=1}^{n} d_i}\right).$$

Theorem 4.7. [24] *Let G be a connected bipartite graph with $n \geq 3$ vertices, m edges and $t(G)$ spanning trees. Then, for any α, $\gamma_2^+ = \gamma_2 \geq \alpha \geq 1$, holds*

$$I_RE(G) = LIE(G) \geq \sqrt{2} + \sqrt{\alpha} + n - 3 + \frac{1}{2} \ln\left(\frac{mt(G)}{\alpha \prod_{i=1}^{n} d_i}\right).$$

Equality holds if and only if $\alpha = 1$ and $G \cong K_{p,q}$, $p + q = n$.

Corollary 4.5. [24] *Let G be a connected bipartite graph with $n \geq 3$ vertices, m edges and $t(G)$ spanning trees. Then,*

$$I_RE(G) = LIE(G) \geq \sqrt{2} + n - 2 + \frac{1}{2} \ln\left(\frac{mt(G)}{\prod_{i=1}^{n} d_i}\right).$$

Equality holds if and only if $G \cong K_{p,q}$, $p + q = n$.

It should be noted that the remaining lower bounds of this section were actually obtained in more general forms in [5, 6, 19].

Theorem 4.8. [5] *Let G be a connected non-bipartite graph with $n \geq 3$ vertices. Then*

$$I_RE(G) \geq \sqrt{2} + \sqrt{n - 2 + (n-1)(n-2)\left(\frac{t(G \times K_2)}{t(G)\prod_{i=1}^{n} d_i}\right)^{1/(n-1)}}.$$

Equality holds if and only if $G \cong K_n$.

Theorem 4.9. [5] *Let G be a connected bipartite graph with $n \geq 3$ vertices, m edges and $t(G)$ spanning trees. Then*

$$I_RE(G) = LIE(G) \geq \sqrt{2} + \sqrt{n - 2 + (n-2)(n-3)\left(\frac{mt(G)}{\prod_{i=1}^{n} d_i}\right)^{1/(n-2)}}.$$

Equality holds if and only if $G \cong K_{p,q}$, $p + q = n$.

Recall that every tree is bipartite . Furthermore, for a tree T, $m = n - 1$ and $t(T) = 1$. Then, from Theorem 4.9, it can be deduced that:

Corollary 4.6. [5] *Let T be a tree with $n \geq 3$ vertices. Then*

$$I_RE(T) = LIE(T) \geq \sqrt{2} + \sqrt{n - 2 + (n-2)(n-3)\left(\frac{n-1}{\prod_{i=1}^{n} d_i}\right)^{1/(n-2)}}.$$

Equality holds if and only if $T \cong K_{1,n-1}$.

Theorem 4.10. [5] *Let G be a connected non-bipartite graph with $n \geq 3$ vertices. Then, there exists a real number $\varepsilon \geq 0$ such that*

$$I_RE(G) \geq \sqrt{2} + (n-1)\left(\frac{t(G \times K_2)}{t(G)\prod_{i=1}^{n} d_i}\right)^{1/2(n-1)} + \varepsilon.$$

Theorem 4.11. [19] *Let G be a connected bipartite graph with $n \geq 3$ vertices, m edges and $t(G)$ spanning trees. Then, there exists a real number $\varepsilon \geq 0$ such that*

$$I_R E(G) = LIE(G) \geq \sqrt{2} + (n-2)\left(\frac{mt(G)}{\prod_{i=1}^{n} d_i}\right)^{1/2(n-2)} + \varepsilon .$$

Theorem 4.12. [6] *Let G be a connected graph with $n \geq 3$ vertices. Then*

$$I_R E(G) \geq \sqrt{2} + (n-2)\sqrt{\frac{n-2}{n+2R_{-1}(G)-4}}. \tag{4.3}$$

Equality holds if and only if $G \cong K_n$.

Remark 4.2. *It is worth noting here that the lower bound* (4.3) *is stronger than the lower bound* (4.1) *for connected graphs. As can be seen in the inequality below*

$$\sqrt{2} + (n-2)\sqrt{\frac{n-2}{n+2R_{-1}(G)-4}} \geq \frac{n}{\sqrt{2}}$$

that is,

$$\sqrt{\frac{2(n-2)}{n+2R_{-1}(G)-4}} \geq 1$$

this implies that

$$R_{-1}(G) \leq \frac{n}{2}$$

which is true for the general Randić index $R_{-1}(G)$, see [20].

Corollary 4.7. [6] *Let G be a connected graph with $n \geq 3$ vertices and minimum degree δ. Then*

$$I_R E(G) \geq \sqrt{2} + (n-2)\sqrt{\frac{n-2}{n\left(1+\frac{1}{\delta}\right)-4}}.$$

Equality holds if and only if $G \cong K_n$

5 Upper Bounds for $I_R E(G)$

In this section, we present some upper bounds on $I_R E(G)$ involving various structural graph parameters.

Theorem 5.1. [8, 14] *Let G be a connected graph of order $n \geq 2$. Then*

$$I_R E(G) \leq \sqrt{2} + \sqrt{(n-1)(n-2)}. \tag{5.1}$$

Equality holds if and only if $G \cong K_n$.

Remark 5.1. [31] *Among all graphs of order n, the empty graph has the minimum $I_RE(G)$ while the complete graph K_n reaches the maximum.*

Theorem 5.2. [8] *Let G, $G \ncong K_n$, be a connected graph of order $n \geq 2$. Then*

$$I_RE(G) \leq \sqrt{2} + 1 + \sqrt{(n-2)(n-3)}\,. \tag{5.2}$$

Equality holds if and only if $\gamma_1^+ = 2$, $\gamma_2^+ = 1$ and $\gamma_i^+ = \frac{n-3}{n-2}$, for $i = 3, \ldots, n$.

Theorem 5.3. [14] *Let G be a bipartite graph of order n with no isolated vertices. Then*

$$I_RE(G) = LIE(G) \leq \sqrt{2} + n - 2\,. \tag{5.3}$$

Equality holds if and only if G is a complete bipartite graph.

Remark 5.2. [14] *Since every tree is a bipartite graph, for any tree T*

$$I_RE(T) = LIE(T) \leq \sqrt{2} + n - 2$$

with equality if and only if $T \cong K_{1,n-1}$. Furthermore, among all trees with n vertices, the star graph $K_{1,n-1}$ is the unique graph with maximum Randić incidence energy.

Theorem 5.4. [24] *Let G be a connected non–bipartite graph with $n \geq 3$ vertices. Then*

$$I_RE(G) \leq \sqrt{2} + \sqrt{\frac{n-1}{2}\left(2(n-2) - \left(\sqrt{\frac{n+2R_{-1}(G)-4}{n-2}} - \sqrt{\frac{n-2R_{-1}(G)}{n}}\right)^2\right)}\,.$$

Equality holds if and only if $G \cong K_n$.

Theorem 5.5. [24] *Let G be a connected non–bipartite graph with $n \geq 3$ vertices. Then, for any α, $\gamma_2^+ \geq \alpha \geq \sqrt{\frac{n+2R_{-1}(G)-4}{n-1}}$, holds*

$$I_RE(G) \leq \sqrt{2} + \sqrt{\alpha} + \left((n-2)^3(n+2R_{-1}(G)-4-\alpha^2)\right)^{1/4}\,.$$

Equality holds if and only if $\alpha = \frac{n-2}{n-1}$ and $G \cong K_n$.

Corollary 5.1. [24] *Let G be a connected non–bipartite graph with $n \geq 3$ vertices. Then*

$$I_RE(G) \leq \sqrt{2} + ((n-1)^3(n+2R_{-1}(G)-4))^{1/4}\,.$$

Equality holds if and only if $G \cong K_n$.

Corollary 5.2. [24] *Let G be a connected non–bipartite graph with $n \geq 3$ vertices and minimum degree δ. Then*

$$I_RE(G) \leq \sqrt{2} + \left((n-1)^3\left(\frac{n(1+\delta)}{\delta} - 4\right)\right)^{1/4}\,.$$

Equality holds if and only if $G \cong K_n$.

Theorem 5.6. [24] *Let G be a connected bipartite graph with $n \geq 3$ vertices. Then, for any α, $\gamma_2^+ = \gamma_2 \geq \alpha \geq \sqrt{\frac{n+2R_{-1}(G)-4}{n-2}}$, holds*

$$I_R E(G) = LIE(G) \leq \sqrt{2} + \sqrt{\alpha} + \left((n-3)^3(n+2R_{-1}(G)-4-\alpha^2)\right)^{1/4}.$$

Equality holds if and only if $\alpha = 1$ and $G \cong K_{p,q}$, $p+q=n$.

Corollary 5.3. [24] *Let G be a connected bipartite graph with $n \geq 3$ vertices. Then*

$$I_R E(G) = LIE(G) \leq \sqrt{2} + \left((n-2)^3(n+2R_{-1}(G)-4)\right)^{1/4}.$$

Equality holds if and only if $G \cong K_{p,q}$, $p+q=n$.

Theorem 5.7. [24] *Let G be a connected non–bipartite graph with $n \geq 3$ vertices. Then, for any α, $\gamma_2^+ \geq \alpha \geq \frac{n-2}{n-1}$, holds*

$$I_R E(G) \leq \sqrt{2} + \sqrt{\alpha} + \sqrt{(n-2)(n-2-\alpha)}. \tag{5.4}$$

Equality holds if and only if $\alpha = \frac{n-2}{n-1}$ and $G \cong K_n$.

Remark 5.3. [24] *The upper bounds* (5.1) *and* (5.2) *are, respectively, obtained from* (5.4) *for $\alpha = \frac{n-2}{n-1}$ and $\alpha = 1$.*

Theorem 5.8. [24] *Let G be a connected bipartite graph with $n \geq 3$ vertices. Then, for any α, $\gamma_2^+ = \gamma_2 \geq \alpha \geq 1$, holds*

$$I_R E(G) = LIE(G) \leq \sqrt{2} + \sqrt{\alpha} + \sqrt{(n-3)(n-2-\alpha)}. \tag{5.5}$$

Equality holds if and only if $\alpha = 1$ and $G \cong K_{p,q}$, $p+q=n$.

Remark 5.4. [24] *Note that the inequality* (5.3) *is obtained from* (5.5) *for $\alpha = 1$.*

It is worth mentioning that the remaining upper bounds of this section were established in more general forms in [5, 6].

Theorem 5.9. [5] *Let G be a connected non-bipartite graph with $n \geq 3$ vertices. Then*

$$I_R E(G) \leq \sqrt{2} + \sqrt{(n-2)^2 + (n-1)\left(\frac{t\,(G \times K_2)}{t\,(G)\prod_{i=1}^{n} d_i}\right)^{1/(n-1)}}. \tag{5.6}$$

Equality holds if and only if $G \cong K_n$.

Remark 5.5. [5] *By using arithmetic-geometric mean inequality, it can be easily shown that the upper bound* (5.6) *is better than the upper bound* (5.1) *for connected non-bipartite graphs.*

Theorem 5.10. [5] *Let G be a connected bipartite graph with $n \geq 3$ vertices, m edges and $t(G)$ spanning trees. Then*

$$I_RE(G) = LIE(G) \leq \sqrt{2} + \sqrt{(n-2)(n-3) + (n-2)\left(\frac{mt(G)}{\prod_{i=1}^{n} d_i}\right)^{1/(n-2)}}. \tag{5.7}$$

Equality holds if and only if $G \cong K_{p,q}$, $p+q=n$.

Remark 5.6. [5] *From arithmetic-geometric mean inequality, it can be seen that the upper bound* (5.7) *is stronger than the upper bound* (5.3) *for connected bipartite graphs. Moreover, the upper bound* (5.3) *was obtained in more general form in Theorem 3.7 of [3].*

For a tree T, $m = n-1$ and $t(T) = 1$. The following result is obvious from (5.7).

Corollary 5.4. [5] *Let T be a tree with $n \geq 3$ vertices. Then*

$$I_RE(T) = LIE(T) \leq \sqrt{2} + \sqrt{(n-2)(n-3) + (n-2)\left(\frac{n-1}{\prod_{i=1}^{n} d_i}\right)^{1/(n-2)}}.$$

Equality holds if and only if $T \cong K_{1,n-1}$.

Theorem 5.11. [6] *Let G be a connected non-bipartite graph with $n \geq 3$ vertices. Then*

$$I_RE(G) \leq \sqrt{2} + \sqrt{1 - \frac{2R_{-1}(G)}{n}} + \sqrt{(n-2)\left(n-3+\frac{2R_{-1}(G)}{n}\right)}. \tag{5.8}$$

Equality holds if and only if $G \cong K_n$.

Theorem 5.12. [6] *Let G be a connected non-bipartite graph with $n \geq 3$ vertices and maximum degree Δ. Then*

$$I_RE(G) \leq \sqrt{2} + \sqrt{1 - \frac{1}{\Delta}} + \sqrt{(n-2)\left(n-3+\frac{1}{\Delta}\right)}. \tag{5.9}$$

Equality holds if and only if $G \cong K_n$.

Remark 5.7. [6] *The upper bounds* (5.8) *and* (5.9) *are better than the upper bound* (5.1) *for connected non-bipartite graphs. Furthermore,* (5.8) *is the best for $I_RE(G)$ among the mentioned upper bounds.*

Theorem 5.13. [6] *Let G be a connected bipartite graph with bipartition $V = X \cup Y$ and $p = |X| > 1$, $q = |Y| > 1$. If $G \cong K_{p,q}$, then $I_RE(G) = LIE(G) = \sqrt{2} + n - 2$ [14]. Otherwise,*

$$I_RE(G) = LIE(G) \leq \sqrt{2} + \sqrt{1 + \frac{1}{\sqrt{pq}}} + \sqrt{1 - \frac{1}{\sqrt{pq}}} + n - 4. \tag{5.10}$$

Equality holds if and only if $G \cong K_{p,q} - e$ (e is any edge in $K_{p,q}$).

Remark 5.8. [6] *Notice that the upper bound* (5.10) *is better than the upper bound* (5.3) *for any connected bipartite graph* G $(\ncong K_{p,q})$ *with bipartition* $V = X \cup Y$ *and* $p = |X| > 1$, $q = |Y| > 1$.

Remark 5.9. [6] *Among all connected bipartite graphs except complete biparite graph,* $K_{p,q} - e$ *has the maximum Randić (normalized) incidence energy or Laplacian incidence energy.*

6 Coulson Integral Formula of $I_R E(G)$

As early as the 1940s [10], Coulson obtained the following integral formula for graph energy

$$E(G) = \frac{1}{\pi}\int_{-\infty}^{+\infty}\left[n - \frac{ix\phi'(A(G),ix)}{\phi(A(G),ix)}\right]dx$$

where $\phi(A(G),x)$ is the characteristic polynomial of the adjacency matrix $A(G)$ of the n-vertex graph G. This fomula is known as *Coulson integral formula* in the literature. Its generalization [25] directly implies the following integral formula for Randić (normalized) incidence energy [8].

Theorem 6.1. [8] *Let* G *be a graph of order* n *and* $\phi(\mathscr{L}^+(G),x)$ *be the characteristic polynomial of its normalized signless Laplacian matrix* $\mathscr{L}^+(G)$. *Then*

$$I_R E(G) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[2n - \frac{ixf'(ix)}{f(ix)}\right]dx$$

where $f(x) = \phi\left(\mathscr{L}^+(G),x^2\right)$.

The coefficient form of the characteristic polynomial of the normalized signless Laplacian matrix $\mathscr{L}^+(G)$ can be expressed as [8]

$$\phi\left(\mathscr{L}^+(G),x\right) = \sum_{k=0}^{n}(-1)^k b_k(G)\,x^{n-k}. \tag{6.1}$$

The another way to write the Coulson integral formula was presented in [8] as follows:

Theorem 6.2. [8] *Let* G *be a graph of order* n *and let* $\phi(\mathscr{L}^+(G),x)$ *be of the form given by* (6.1). *Then*

$$I_R E(G) = \frac{1}{\pi}\int_{0}^{+\infty}\ln\left(\sum_{k=0}^{n} b_k(G)\,x^{2k}\right)\frac{dx}{x^2}.$$

In [8], It was also pointed out that the above result makes it possible to compare the Randić (normalized) incidence energies of two graphs.

Corollary 6.1. [8] *Let* G_1 *and* G_2 *be two* n*-vertex graphs. If* $b_k(G_1) \le b_k(G_2)$ *for* $0 \le k \le n$, *then* $I_R E(G_1) \le I_R E(G_2)$. *Moreover, if a strict inequality* $b_k(G_1) < b_k(G_2)$ *holds for some* $0 \le k \le n$, *then* $I_R E(G_1) < I_R E(G_2)$.

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