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Remark on the upper bounds for arithmetic–geometric topological index of graphs

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Abstract: In this paper two new inequalities involving upper bounds for arithmetic–geometric topological index are obtained.

Keywords: Topological indices, arithmetic–geometric index.

1 Introduction

Let $G = (V,E)$, $V = \{v_1, v_2, \dots, v_n\}$, $E = \{e_1, e_2, \dots, e_m\}$, be a simple connected graph of order n and size m. DEnote by $\Delta = d_1 \geq d_2 \geq \cdots \geq d_n = \delta > 0$, $d_i = d(v_i)$, and $\Delta_e = d(e_1)+2 \geq d(e_2)+2 \geq \cdots \geq d(e_m)+2 = \delta_e > 0$ the sequences of its vertex and edge degrees, respectively, given in an nonincreasing order. If vertices v_i and v_j are adjacent in G, we write $i \sim j$. As usual, $L(G)$ denotes a line graph of graph G.

Let v_j and v_k be two nonadjacent vertices in G that are both adjacent to v_i. Denote by $\Gamma(G)$ a class of connected graphs with the property $d_i = \sqrt{d_j d_k}$, or $L(G)$ is bidegreed.

A topological index, or a graph invariant, is a numerical quantity which is invariant under automorphisms of the graph. Many of them are defined as simple functions of the degrees of the vertices and edges, distances between vertices, or various graph spectra. Here we recall definitions of some vertex–degree-based indices that are of interest for the present paper.

The first Zagreb index, $M_1(G)$, introduced in [6], is defined as

$$M_1(G) = \sum_{i=1}^{n} d_i^2.$$

In [4] it is proven that the first Zagreb index can also be expressed as

$$M_1 = \sum_{i\sim j} (d_i + d_j).$$

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The second Zagreb index, $M_2(G)$, is defined as [7]

$$M_2(G) = \sum_{i\sim j} d_i d_j \,,$$

and the so called general Randić index, $R_{-1}(G)$, as [2]

$$R_{-1}(G) = \sum_{i\sim j} \frac{1}{d_i d_j}.$$

The geometric–arithmetic index, $GA(G)$ index for short, proposed in [28], is defined to be

$$GA(G) = \sum_{i\sim j} \frac{2\sqrt{d_i d_j}}{d_i + d_j}.$$

By analogy with GA index, the arithmetic–geometric index, $AG(G)$, is defined as [25]

$$AG(G) = \sum_{i\sim j} \frac{d_i + d_j}{2\sqrt{d_i d_j}}.$$

More on this topological index can be found, for example, in [12, 17, 21, 24, 27].

In this paper we prove two inequalities involving upper bounds for $AG(G)$ index, and obtain many upper bounds for $AG(G)$ reported in the literature as corollaries.

2 Preliminaries

In this section we state one analytical inequality for real number sequences which will be used in the rest of the paper.

Lemma 2.1. *[23] Let $x = (x_i)$, $i = 1, 2, \dots, n$, be a sequence of nonnegative real numbers and $a = (a_i)$, $i = 1, 2, \dots, n$, be positive real number sequence. Then for any $r \geq 0$ holds*

$$\sum_{i=1}^{n} \frac{x_i^{r+1}}{a_i^r} \geq \frac{\left(\sum_{i=1}^{n} x_i\right)^{r+1}}{\left(\sum_{i=1}^{n} a_i\right)^{r}}. \tag{2.1}$$

Equality holds if and only if $r = 0$, or $\frac{x_1}{a_1} = \frac{x_2}{a_2} = \cdots = \frac{x_n}{a_n}$.

3 Main results

In the next theorem we establish relation between $AG(G)$ and $R_{-1}(G)$.

Theorem 3.1. *Let G be a simple connected graph with $n \geq 3$ vertices and m edges. Then*

$$AG(G) \leq \frac{1}{2}\sqrt{m(n(\Delta_e+\delta_e)-\Delta_e\delta_e R_{-1}(G))}. \tag{3.1}$$

Equality holds if and only if G is a regular or semiregular bipartite graph or $G \in \Gamma(G)$.

Proof. For any two adjacent vertices v_i and v_j in G we have that

$$(d_i+d_j-\Delta_e)(d_i+d_j-\delta_e) \leq 0,$$

i.e.

$$(d_i+d_j)^2+\delta_e\Delta_e \leq (\Delta_e+\delta_e)(d_i+d_j). \tag{3.2}$$

After dividing the above inequality by $4d_id_j$ and summing over all edges in G, we get

$$\sum_{i\sim j}\frac{(d_i+d_j)^2}{4d_id_j}+\frac{\Delta_e\delta_e}{4}\sum_{i\sim j}\frac{1}{d_id_j} \leq \frac{\Delta_e+\delta_e}{4}\sum_{i\sim j}\frac{d_i+d_j}{d_id_j},$$

that is

$$\sum_{i\sim j}\frac{(d_i+d_j)^2}{4d_id_j} \leq \frac{1}{4}\left(n(\Delta_e+\delta_e)-\Delta_e\delta_e R_{-1}(G)\right). \tag{3.3}$$

On the other hand, for $r=1$, $x_i := \frac{d_i+d_j}{2\sqrt{d_id_j}}$, $a_i := 1$, with summation performed over all edges in graph G, the inequality (2.1) becomes

$$\sum_{i\sim j}\frac{(d_i+d_j)^2}{4d_id_j} \geq \frac{\left(\sum\limits_{i\sim j}\frac{d_i+d_j}{2\sqrt{d_id_j}}\right)^2}{\sum\limits_{i\sim j}1},$$

that is

$$\sum_{i\sim j}\frac{(d_i+d_j)^2}{4d_id_j} \geq \frac{AG(G)^2}{m}. \tag{3.4}$$

Now, from (3.3) and (3.4) it follows

$$\frac{AG(G)^2}{m} \leq \frac{1}{4}\left(n(\Delta_e+\delta_e)-\Delta_e\delta_e R_{-1}(G)\right),$$

from which (3.1) is obtained.

Equality in (3.2) holds if and only if $d_i+d_j \in \{\Delta_e, \delta_e\}$ for any pair of adjacent vertices v_i and v_j in G. This means that equality in (3.2) holds if and only if $L(G)$ is regular or bidegreed graph.

Equality in (3.3) holds if and only if $\frac{d_i+d_j}{2\sqrt{d_id_j}}$ is constant for any pair of adjacent vertices v_i and v_j in G. Let v_j and v_k be two vertices adjacent to v_i. Then

$$\frac{d_i+d_j}{\sqrt{d_id_j}} = \frac{d_i+d_k}{\sqrt{d_id_k}},$$

i.e.

$$\left(\sqrt{d_k}-\sqrt{d_j}\right)\left(d_i-\sqrt{d_jd_k}\right)=0.$$

If $d_k = d_j$, then equality in (3.3) holds if and only if G is regular or semiregular bipartite graph. Suppose that $d_k \neq d_j$. Then equality in (3.3) holds if and only if $d_i = \sqrt{d_jd_k}$. This means that equality in (3.3), and consequently in (3.1), holds if and only if G is a regular or semiregular bipartite graph or $G \in \Gamma(G)$. □

Corollary 3.1. *Let G be a simple connected graph with $n \geq 3$ vertices and m edges. Then*

$$AG(G) \leq \frac{n}{4}\left(\sqrt{\frac{\Delta_e}{\delta_e}}+\sqrt{\frac{\delta_e}{\Delta_e}}\right)\sqrt{\frac{m}{R_{-1}(G)}}. \tag{3.5}$$

Equality holds if and only if G is regular or semiregular bipartite graph or $G \in \Gamma(G)$ with even number of vertices.

Proof. Based on the arithmetic–geometric mean inequality, AM–GM (see for example [20]), according to (3.1) we have that

$$2\sqrt{4m\Delta_e\delta_eR_{-1}(G)AG(G)^2} \leq 4AG(G)^2+m\Delta_e\delta_eR_{-1}(G) \leq mn(\Delta_e+\delta_e),$$

from which (3.5) is obtained. □

Since

$$2\delta \leq \delta_e \leq d_i+d_j \leq \Delta_e \leq 2\Delta,$$

for every edge in G, with equalities holding if and only if G is regular, we have the following corollaries of Theorem 3.1.

Corollary 3.2. *Let G be a simple connected graph with $n \geq 3$ vertices and m edges. Then*

$$AG(G) \leq \frac{\sqrt{2}}{2}\sqrt{m(n(\Delta+\delta)-2\Delta\delta R_{-1}(G))}.$$

Equality holds if and only if G is a regular graph.

Corollary 3.3. *Let G be a simple connected graph with $n \geq 3$ vertices and m edges. Then*

$$AG(G) \leq \frac{n}{4}\left(\sqrt{\frac{\Delta}{\delta}}+\sqrt{\frac{\delta}{\Delta}}\right)\sqrt{\frac{m}{R_{-1}(G)}}. \tag{3.6}$$

Equality holds if and only if G is a regular graph.

Corollary 3.4. *Let G be a simple connected graph with $n \geq 3$ vertices and m edges. Then*

$$AG(G) \leq \frac{n}{4}\left(\sqrt{\frac{\Delta}{\delta}}+\sqrt{\frac{\delta}{\Delta}}\right)\sqrt{\frac{M_2(G)}{m}}. \tag{3.7}$$

Equality holds if and only if G is regular.

Proof. From the arithmetic–harmonic mean inequality, AM–HM (see for example [20]), we get

$$\sum_{i\sim j} d_i d_j \sum_{i\sim j} \frac{1}{d_i d_j} \geq m^2,$$

i.e.

$$\frac{m}{R_{-1}(G)} \leq \frac{M_2(G)}{m}. \tag{3.8}$$

From the above and (3.6) we get (3.7). □

In the following theorem we prove the inequality involving an upper bound for $AG(G)$ in terms of $M_1(G)$ and parameter n.

Theorem 3.2. *Let G be a simple connected graph with $n \geq 2$ vertices. Then*

$$AG(G) \leq \frac{1}{2}\sqrt{nM_1(G)}. \tag{3.9}$$

Equality holds if and only if G is a regular or semiregular bipartite graph.

Proof. For $r = 1$, $x_i := \frac{d_i+d_j}{\sqrt{d_i d_j}}$, $a_i := d_i + d_j$, with summation performed over all edges in graph G, the inequality (2.1) transforms into

$$\sum_{i\sim j} \frac{(d_i+d_j)^2}{d_i d_j (d_i+d_j)} = \sum_{i\sim j} \frac{\left(\frac{d_i+d_j}{2\sqrt{d_i d_j}}\right)^2}{\frac{1}{4}(d_i+d_j)} \geq \frac{\left(\sum_{i\sim j} \frac{d_i+d_j}{2\sqrt{d_i d_j}}\right)^2}{\sum_{i\sim j} \frac{1}{4}(d_i+d_j)},$$

that is

$$\sum_{i\sim j} \frac{d_i+d_j}{d_i d_j} \geq \frac{AG(G)^2}{\frac{1}{4}M_1(G)}. \tag{3.10}$$

Since

$$\sum_{i\sim j} \frac{d_i+d_j}{d_i d_j} = \sum_{i\sim j} \left(\frac{1}{d_i} + \frac{1}{d_j}\right) = \sum_{i=1}^{n} d_i \cdot \frac{1}{d_i} = n,$$

from (3.10) it follows

$$n \geq \frac{AG(G)^2}{\frac{1}{4}M_1(G)},$$

from which (3.9) is obtained.

Equality in (3.10) holds if and only if $\frac{1}{\sqrt{d_i d_j}}$ is constant for any pair of adjacent vertices v_i and v_j in G. Let v_j and v_k be two vertices adjacent to v_i. Then

$$\frac{1}{\sqrt{d_i d_j}} = \frac{1}{\sqrt{d_i d_k}},$$

i.e. $d_j = d_k$. This means that equality in (3.10), and therefore in (3.9), holds if and only if G is a regular or semiregular bipartite graph. □

As mentioned, upper bound for $AG(G)$ given by (3.9) depends on $M_1(G)$. Since the first Zagreb index is for sure the most examined topological index in the literature (see, for example, [1, 8, 9, 22] and the literature cited therein), the inequality (3.9) enables to determine upper bounds for $AG(G)$ in terms of many different graph parameters. In the following corollaries we give some of them.

Corollary 3.5. *Let G be a simple connected graph with n vertices and m edges. Then*

$$AG(G) \leq \frac{1}{2}\sqrt{n(2m(\Delta+\delta)-n\Delta\delta)}. \tag{3.11}$$

Equality holds if and only if G is regular or semiregular bipartite graph.

Proof. In [3] (see also [10, 11, 13, 14, 19]) it was proven that

$$M_1(G) \leq 2m(\Delta+\delta)-n\Delta\delta,$$

with equality holding if and only if $d_i \in \{\Delta, \delta\}$ for every $i = 1, 2, \dots, n$. From the above and (3.9) we arrive at (3.11). □

Corollary 3.6. *Let G be a simple connected graph with $m \geq 1$ edges. Then*

$$AG(G) \leq \frac{m}{2}\left(\sqrt{\frac{\Delta}{\delta}}+\sqrt{\frac{\delta}{\Delta}}\right). \tag{3.12}$$

Equality holds if and only if G is a regular or biregular graph.

Proof. In [15] (see also [5, 11, 26]) it was proven that

$$M_1(G) \leq \frac{m^2}{n}\left(\sqrt{\frac{\Delta}{\delta}}+\sqrt{\frac{\delta}{\Delta}}\right)^2.$$

From the above and (3.9) we arrive at (3.12). □

The inequality (3.12) was proven in [17] (see also [24]).

Since

$$M_1(G) \leq m\Delta_e \leq 2m\Delta \leq n\Delta^2 \leq n(n-1)^2,$$

we have the following corollary of Theorem 3.2.

Corollary 3.7. *Let G be a simple connected graph with $n \geq 2$ vertices and m edges. Then*

$$AG(G) \leq \frac{1}{2}\sqrt{nm\Delta_e} \leq \frac{1}{2}\sqrt{2nm\Delta} \leq \frac{n\Delta}{2} \leq \frac{n(n-1)}{2}. \tag{3.13}$$

The last two inequalities in (3.13) were proven in [27].

Corollary 3.8. *Let G be a simple connected graph with $n \geq 2$ vertices and m edges. Then*

$$AG(G) \leq \frac{1}{2}\sqrt{4m^2 + n^2\alpha(n)(\Delta-\delta)^2}, \tag{3.14}$$

where

$$\alpha(n) = \frac{1}{4}\left(1 - \frac{1+(-1)^{n+1}}{2n^2}\right).$$

Equality holds if and only if G is a regular graph.

Proof. In [18] the following inequality was proven

$$M_1(G) \leq \frac{4m^2}{n} + n\alpha(n)(\Delta-\delta)^2.$$

From the above and (3.9) we obtain (3.14). □

Remark 3.1. *Since $\alpha(n) \leq \frac{1}{4}$ for every n, we have that*

$$M_1(G) \leq \frac{4m^2}{n} + \frac{n}{4}(\Delta-\delta)^2,$$

which was proven in [5]. According to this, it follows

$$AG(G) \leq \frac{1}{2}\sqrt{4m^2 + \frac{n^2}{4}(\Delta-\delta)^2}.$$

Corollary 3.9. *Let T be a tree with $n \geq 2$ vertices. Then*

$$AG(T) \leq \frac{1}{2}\sqrt{n(2(n-1)+(n-2)\Delta)}. \tag{3.15}$$

Equality holds if and only if $T \cong K_{1,n-1}$.

Proof. In [16] the following inequality was proven

$$M_1(T) \leq 2(n-1)+(n-2)\Delta.$$

From the above and (3.9) we arrive at (3.15). □

Since $\Delta \leq n-1$, we have the following corollary.

Corollary 3.10. *Let T be a tree with $n \geq 2$ vertices. Then*

$$AG(T) \leq \frac{n\sqrt{n-1}}{2}. \tag{3.16}$$

Equality holds if and only if $T \cong K_{1,n-1}$.

The inequality (3.16) was proven in [27].

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References

[1] B. BOROVIĆANIN, K. C. DAS, B. FURTULA, I. GUTMAN, *Bounds for Zagreb indices*, MATCH Commun. Math. Comput. Chem. 78 (2017) 17–100.

[2] M. CAVERS, S. FALLAT, S. KIRKLAND, *On the normalized Laplacian energy and general Randić index R_{-1} of graphs*, Lin. Algebra Appl. 433(1) (2010) 172–190.

[3] K. C. DAS, *Maximizing the sum of the squares of the degrees of a graph*, Discrete Math. 285 (2004) 57–66.

[4] T. DOŠLIĆ, B. FURTULA, A. GRAOVAC, I. GUTMAN, S. MORADI, Z. YARAHMADI, *On vertex-degree-based molecular structure descriptors*, MATCH Commun. Math. Comput. Chem. 66 (2011) 613–626.

[5] G. H. FATH-TABAR, *Old and new Zagreb indices of graphs*, MATCH Commun. Math. Comput. Chem. 65 (2011) 79–84.

[6] I. GUTMAN, N. TRINAJSTIĆ, *Graph theory and molecular orbitals. Total π-electron energy of alternant hydrocarbons*, Chem. Phys. Lett. 17 (1972) 535–538.

[7] I. GUTMAN, B. RUŠČIĆ, N. TRINAJSTIĆ, C. F. WILCOX, *Graph theory and molecular orbitals. XII. Acyclic polyenes*, J. Chem. Phys. 62 (1975) 3399–3405.

[8] I. GUTMAN, K. C. DAS, *The first Zagreb index 30 years after*, MATCH Commun. Math. Comput. Chem. 50 (2004) 83–92.

[9] I. GUTMAN, E. MILOVANOVIĆ, I. MILOVANOVIĆ, *Beyond the Zagreb indices*, AKCE Int. J. Graph Comb. 17(1) (2020) 74–85.

[10] S. HOSAMANI, B. BASAVANGOUD, *New upper bounds for the first Zagreb index*, MATCH Commun. Math. Comput. Chem. 74 (2015) 173–182.

[11] A. ILIĆ, M. ILIĆ, B. LIU, *On the upper bounds for the first Zagreb index*, Kragujevac J. Math. 35(1) (2011) 173–182.

[12] Ž. KOVIJANIĆ VUKIĆEVIĆ, S. VUJOŠEVIĆ, G. POPIVODA, *Unicyclic graphs with extremal values of arithmetic–geometric index*, Discr. Appl. Math. 302 (2021) 67–75.

[13] J. LI, W. C. SHIU, A. CHONG, *On the Laplacian Estrada index of a graph*, Appl. Anal. Discr. Math. 3 (2009) 147–156.

[14] X. LI, H. ZHAO, *Trees with the first smallest and largest genaralized topological indices*, MATCH Commun. Math. Comput. Chem. 50 (2005) 57–62.

[15] M. LIU, B. LIU, *New sharp upper bounds for the first Zagreb index*, MATCH Commun. Math. Comput. Chem. 62 (2009) 689–698.

[16] M. MATEJIĆ, E. MILOVANOVIĆ, I. MILOVANOVIĆ, R. KHOEILAR, *A note on the first Zagreb index and coindex of graphs*, Commun. Comb. Optim. 6(1) (2021) 41–51.

[17] I. Ž. MILOVANOVIĆ, M. M. MATEJIĆ, E. I. MILOVANOVIĆ, *Upper bounds for arithmetic–geometric index of graphs*, Sci. Pub. State Univ. Novi Pazar, Ser A: Appl. Math. Inform. Mech. 10(1) (2018) 49–54.

[18] E. I. MILOVANOVIĆ, I. Ž. MILOVANOVIĆ, *Sharp bounds for the first Zagreb index and first Zagreb coindex*, Miskolc Math. Notes 16(2) (2015) 1017–1024.

[19] I. Ž. MILOVANOVIĆ, V. M. ĆIRIĆ, I. Z. MILENTIJEVIĆ, E. I. MILOVANOVIĆ, *On some spectral vertex and edge degree–based graph invariants*, MATCH Commun. Math. Comput. Chem. 77 (2017) 177–188.

[20] D. S. MITRINOVIĆ, P. M. VASIĆ, Analytic inequalities, Springer Verlag, Berlin-Heidelberg-New York, 1970.

[21] E. D. MOLINA, J. M. RODRIGUEZ, J. L. SANCHEZ, J. M. SIGARRETA, *Some properties of the arithmetic–geometric index*, Symmetry 13(5) (2021) 857.

[22] S. NIKOLIĆ, G. KOVAČEVIĆ, A. MILIČEVIĆ, N. TRINAJSTIĆ, *The Zagreb indices 30 years after*, Croat. Chem. Acta **76** (2003) 113–124.

[23] J. RADON, *Über die absolut additiven Mengenfunktionen*, Wiener Sitzungsber. 122 (1913) 1295–1438.

[24] J. M. RODRIGUEZ, J. L. SANCHEZ, J. M. SIGARRETA, E. TOARIS, *Bounds on the arithmetic–geometric index*, Symmetry 13(4) (2021) 689.

[25] V. S. SHEGEHALLI, R. KANABUR, *Arithmetic–geometric indices of path graph*, J. Comput. Math. Sci. 6(1) (2015) 19–24.

[26] G. TION, T. HUANG, S. CUI, *Bounds of the algebraic connectivity of graphs*, Adv. Math. 41(2) (2012) 217–224.

[27] S. VUJOŠEVIĆ, G. POPIVODA, Ž. KOVIJANIĆ VUKIĆEVIĆ, B. FURTULA, R. ŠKREKOVSKI, *Arithmetic–geometric index and its relations with geometric–arithmetic index*, Appl. Math. Comput. 391 (2021) 125706.

[28] D. VUKIČEVIĆ, B. FURTULA, *Topological index based on the ratios of geometrical and arithmetical means of end–vertex degrees of edges*, J. Math. Chem. 46 (2009) 1369–1376.