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The Inner Aggregation Newton's Method for Solving Nonlinear Equations

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Abstract: This paper deals with the new variants of Newton's method based on aggregation functions for finding simple real roots of nonlinear equations. Unlike some well-known two-step modifications of Newton's method based on various means, the presented methods use the same means in a different manner to achieve the third convergence order. A new general iterative scheme is analyzed in details and the theoretical results are verified on several test examples from real-life and literature.

Keywords: Newton's method, aggregation function, order of convergence

1 Introduction

We consider the problem of numerical determination of an exact root α of nonlinear equation

$$f(x) = 0, \quad f : I \to \mathbb{R}, \tag{1.1}$$

where I is some open interval in $\mathbb{R}$, while $\alpha \in \mathbb{R}$ is the simple root, i.e. $f(\alpha) = 0$ and $f'(\alpha) \neq 0$. In general, it is hard or even impossible to find the exact root α analytically, so numerical iterative methods are frequently used to find the root by creating a sequence of approximations $\{x_n\}$ that tends towards α. If there exist finite $C \neq 0$ and $q \geq 1$ such that

$$\lim_{n\to\infty} \frac{x_{n+1} - \alpha}{(x_n - \alpha)^q} = C, \tag{1.2}$$

then the *order of convergence* of the sequence $\{x_n\}$ is equal q, and C is known as the *asymptotic error constant*. For $e_n = x_n - \alpha$, the expression (1.2) can be rewritten as

$$e_{n+1} = Ce_n^q + \mathscr{O}(e_n^{q+1})$$

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which is called the *error equation*.

Probably the best known root-finding iterative method is Newton's method (shortly N method) given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, \ldots \tag{1.3}$$

It is quadratically convergent to simple root if the initial approximation is sufficiently close to α. The corresponding error equation has a form

$$e_{n+1} = \frac{f''(\alpha)}{2f'(\alpha)} \cdot e_n^2 + \mathcal{O}(e_n^3).$$

There is a vast number of multistep iterative methods with the Newton's step as the first step. Those methods have been designed with the aim of increasing the efficiency and the convergence order of method (1.3). For example, Weerakoon and Fernando [17] established a two-point iterative scheme

$$\begin{aligned} \tilde{x}_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= x_n - \frac{f(x_n)}{\frac{1}{2}(f'(x_n) + f'(\tilde{x}_n))}. \end{aligned} \tag{1.4}$$

This method uses the arithmetic mean of $f'(x_n)$ and $f'(\tilde{x}_n)$ in the second step that provides the third convergence order. Özban [12], Lukić and Ralević [10], Ralević and Lukić [14] have replaced the arithmetic mean by harmonic, geometric and root-power means, respectively, to get additional third-order methods. These types of methods are also known as the mean-based methods [1–3, 5, 6].

Recently, Paunović, Ćebić and Ralević [13] have considered a more generalized version of the mean-based third-order methods, defined by

$$\begin{aligned} \tilde{x}_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= x_n - \frac{f(x_n)}{A(f'(x_n), f'(\tilde{x}_n))}, \end{aligned} \tag{1.5}$$

where $A(\cdot,\cdot)$ is an aggregation function. The method is called the external aggregation Newton's method. Contrary to this approach, in order to construct a new family of third-order methods, we use the aggregation function with different arguments (namely, x_n and $\tilde{x}_n$ instead of $f'(x_n)$ and $f'(\tilde{x}_n)$) in the second step of iterative scheme.

A brief recall of the (quasi) aggregation functions is given in the next section. The new family of methods is described in Section 3 with corresponding theoretical analysis. Section 4 is devoted to the numerical examples and concluding remarks.

2 A Quasi Aggregation Function

In this section we present a brief recall of definitions, examples and properties of aggregation functions. Let I be interval real number.

Definition 1. *An $n-ary$* **aggregation function** *is a function $A_{[n]}: I^n \to I$ such that*

i) $A_{[n]}(x_1,...,x_n) \le A_{[n]}(y_1,...,y_n)$ *whenever* $x_i \le y_i$ *for all* $i \in \{1,...,n\}$ *(A is monotonically increasing function in all its arguments).*

ii) $\inf\limits_{x\in I^n} A_{[n]}(x) = \inf I$ *and* $\sup\limits_{x\in I^n} A_{[n]}(x) = \sup I$, $x = (x_1,...,x_n)$ *(boundary condition).*

An **aggregation function** *is a function* $A: \bigcup\limits_{n\in\mathbb{N}} I^n \to I$ *such that for* $n = 1$ *holds* $A(x) = x$, *for all* $x \in I$*, whose restriction is* $A|_{I^n} = A_{[n]}$*, for any* $n \in \mathbb{N}$.

For aggregation function is required to comply with additional features such as

iii) $A_{[n]}(x,x,...,x) = x$ for all $x \in I$ ($A_{[n]}$ is idempotent function).

iv) $A_{[n]}(x_1,...,x_n) = A_{[n]}(x_{p_1},...,x_{p_n})$ for any permutation $(p_1,...,p_n)$ of set $\{1,...,n\}$ ($A_{[n]}$ is symmetric function in all its arguments).

v) $A_{[n]}$ is continuous function.

vi) $A_{[n]} \in C^\ell(I^\circ)$ (The function $A_{[n]}$ has continuous derivatives up to the order ℓ in all variables).

Remark 1. *It is customary to take the* $I = [0,1]$. *In this case boundary condition reduces to* $A(0,...,0) = 0$ *and* $A(1,...,1) = 1$.

Here are some examples that will be used in our research, in the case $n = 2$, when the aggregation function is defined in the following way:

Arithmetic mean

$$A(a_1,a_2) = \frac{a_1+a_2}{2}.$$

Harmonic mean

$$H(a_1,a_2) = \frac{2}{\frac{1}{a_1}+\frac{1}{a_2}}.$$

Geometric mean

$$G(a_1,a_2) = \sqrt{a_1a_2}.$$

Root-power means

$$M_2(a_1,a_2) = \left(\frac{a_1^p+a_2^p}{2}\right)^{1/p}, \quad (p \neq 0).$$

Marginal, i.e. boundary members of these classes are $M_0 = G = M_{\log a}$, which is the geometric mean, while M_∞ =max and $M_{-\infty}$ =min which are not in class of quasi-arithmetic means. For $p = 2$ and $p = 3$ we have:

Quadratic mean

$$Q(a_1,a_2) = \sqrt{\frac{a_1^2 + a_2^2}{2}}.$$

Cubic mean

$$C(a_1,a_2) = \sqrt[3]{\frac{a_1^3 + a_2^3}{2}}.$$

Contraharmonic mean

$$CH(a_1,a_2) = \frac{a_1^2 + a_2^2}{a_1 + a_2}.$$

Heinz means

$$Z_p(a,b) = \frac{a^p b^{1-p} + a^{1-p} b^p}{2}, \quad (0 \leqslant p \leqslant \frac{1}{2}).$$

Generalized Heron means

$$H_p(a_1,a_2) = \Big(\frac{a_1^p + (a_1 a_2)^{p/2} + a_2^p}{3}\Big)^{1/p}, \quad (p \neq 0).$$

Symmetric means

$$Q_p(a_1,a_2) = \frac{a_1^s a_2^t + a_1^t a_2^s}{2}, \quad \Big(s = \frac{1+\sqrt{p}}{2},\ t = \frac{1-\sqrt{p}}{2}\Big).$$

Generalized Contraharmonic means

$$C_p(a_1,a_2) = \frac{a_1^p + a_2^p}{a_1^{p-1} + a_2^{p-1}}.$$

Thus, we shall present a definition, some examples and properties of aggregation functions (see [4,7,9]).

Here we consider the aggregation functions as functions of two real variables.

Definition 2. A quasi aggregation function of class ℓ *is a function* $A : I^n \to I$, $n \in \mathbb{N}$ (*where* I *is the interval of real numbers*) *such that* A *is idempotent, symmetric and* $A \in C^\ell(I^\circ)$, $\ell \in \mathbb{N}$.

Lemma 1. *If* $A : I^n \to I$, $n \in \mathbb{N}$ *is a quasi aggregation function of class* ℓ, *then*

i) $A_{x_i}(x,x,...,x) = A_{x_j}(x,x,...,x)$, $x \in I$, $i,j = 1,...,n$.

ii) $A_{x_i}(x,x,...,x) = \frac{1}{n}$, $x \in I$, $i, j = 1,...,n$,

where A_{x_i} *denotes* $\frac{\partial A}{\partial x_i}$.

Proof. i) The function A is symmetric and $A \in C^{\ell}(I)$. Thus, if for example $i = 1$ and $j = 2$

$$\frac{\partial A}{\partial x_1}(x,x,...,x) = \lim_{h\to 0} \frac{A(x+h,x,...,x) - A(x,x,...,x)}{h}$$
$$= \lim_{h\to 0} \frac{A(x,x+h,...,x) - A(x,x,...,x)}{h} = \frac{\partial A}{\partial x_2}(x,x,...,x).$$

ii) Differentiating the equality $A(x,x,...,x) = x$, $x \in I$, we obtain

$$A_{x_1}(x,x,...,x)\cdot 1 + A_{x_2}(x,x,...,x)\cdot 1 + ... + A_{x_n}(x,x,...,x)\cdot 1 = 1,$$

i.e. $A_{x_i}(x,x,...,x) = \frac{1}{n}$.

□

Note that, for $n = 2$, $A_x(x,x) = A_y(x,x) = \frac{1}{2}$.

Lemma 2. *If* $A : \bigcup_{n\in\mathbb{N}} I^n \to I$ *is a quasi aggregation function of class* ℓ*, then holds:*

i) $A_x(x,y) = A_y(y,x)$;

ii) $A_{xx}(x,y) = A_{yy}(y,x)$;

iii) $A_{xy}(x,y) = A_{yx}(y,x)$;

iv) $A_{xy}(x,y) = A_{xy}(y,x)$ *(*A_{xy} *is symmetric function);*

v) $A_{xxy}(x,y) = A_{yyx}(y,x)$;

vi) $A_{xxx}(x,y) = A_{yyy}(y,x)$.

Proof. i) $A_x(x,y) = \lim_{h\to 0} \frac{A(x+h,y)-A(x,y)}{h} = \lim_{h\to 0} \frac{A(y,x+h)-A(y,x)}{h} = A_y(y,x)$.

ii) $A_{xx}(x,y) = \lim_{h\to 0} \frac{A_x(x+h,y)-A_x(x,y)}{h} = \lim_{h\to 0} \frac{A_y(y,x+h)-A_y(y,x)}{h} = A_{yy}(y,x)$.

iii) $A_{xy}(x,y) = \lim_{h\to 0} \frac{A_x(x,y+h)-A_x(x,y)}{h} = \lim_{h\to 0} \frac{A_y(y+h,x)-A_y(y,x)}{h} = A_{yx}(y,x)$.

iv) A_{xy} and A_{xy} are continuous function, that is $A_{xy}(u,v) = A_{xy}(u,v)$:
$A_{xy}(x,y) = A_{yx}(y,x) = A_{xy}(y,x)$.

v) $A_{xxy}(x,y) = \lim_{h\to 0} \frac{A_{xx}(x,y+h)-A_{xx}(x,y)}{h} = \lim_{h\to 0} \frac{A_{yy}(y+h,x)-A_{yy}(y,x)}{h} = A_{yyx}(y,x)$.

vi) $A_{xxx}(x,y) = \lim_{h\to 0} \frac{A_{xx}(x+h,y)-A_{xx}(x,y)}{h} = \lim_{h\to 0} \frac{A_{yy}(y,x+h)-A_{yy}(y,x)}{h} = A_{yyy}(y,x)$.

□

3 Definition of the Method and Analysis of its Convergence

Let A be quasi aggregation function of class 3. Then we can define **the inner aggregation Newton's method** (shortly IANM) with the following iterative scheme

$$\begin{aligned} \tilde{x}_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= x_n - \frac{f(x_n)}{f'(A(x_n,\tilde{x}_n))}. \end{aligned} \tag{3.1}$$

Theorem 1. *Suppose that $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is sufficiently differentiable function in open interval I and f has a simple root $\alpha \in I$. If x_0 is chosen sufficiently close to α, then IANM possesses cubic convergence and it satisfies the following error equation*

$$e_{n+1} = \big(c_2 A_{xx}(\alpha,\alpha) - \frac{1}{4}c_3\big)e_n^3 + o(e_n^3), \tag{3.2}$$

where $c_j = \frac{f^{(j)}(\alpha)}{j!f'(\alpha)}$ for $j = 2,3,\dots$

Proof. From Taylor's expansion of $f(x)$ about α, we get

$$\begin{aligned} f(x_n) &= f(\alpha) + f'(\alpha)e_n + \tfrac{1}{2!}f''(\alpha)e_n^2 + \tfrac{1}{3!}f^{(3)}(\alpha)e_n^3 + o(e_n^3) \\ &= f'(\alpha)[e_n + c_2e_n^2 + c_3e_n^3 + o(e_n^3)], \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} f'(x_n) &= f'(\alpha) + f''(\alpha)e_n + \tfrac{1}{2!}f'''(\alpha)e_n^2 + \tfrac{1}{3!}f^{(4)}(\alpha)e_n^3 + o(e_n^3) \\ &= f'(\alpha)[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + o(e_n^3)]. \end{aligned} \tag{3.4}$$

Dividing (3.3) by (3.4), we have

$$\begin{aligned} \frac{f(x_n)}{f'(x_n)} &= [e_n + c_2e_n^2 + c_3e_n^3 + o(e_n^3)][1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + o(e_n^3)]^{-1} \\ &= [e_n + c_2e_n^2 + c_3e_n^3 + o(e_n^3)][1 - 2c_2e_n + (4c_2^2 - 3c_3)e_n^2 + o(e_n^2)] \\ &= e_n - c_2e_n^2 + 2(c_2^2 - c_3)e_n^3 + o(e_n^3), \end{aligned}$$

so, from the first step of iterative scheme, it is easy to get

$$\tilde{x}_n = x_n - \frac{f(x_n)}{f'(x_n)} = \alpha + c_2e_n^2 + 2(c_3 - c_2^2)e_n^3 + o(e_n^3). \tag{3.5}$$

Hence, from (3.4) and (3.5) we have

$$\begin{aligned} f'(\tilde{x}_n) &= f'(\alpha) + f''(\alpha)(\tilde{x}_n - \alpha) + o(\tilde{x}_n - \alpha) \\ &= f'(\alpha)[1 + 2c_2^2e_n^2 + 4(c_2c_3 - c_2^3)e_n^3 + o(e_n^3)]. \end{aligned} \tag{3.6}$$

Let Ω be an open set in $\mathbb{R}^2$, $(x_0,y_0)\in\Omega$, and let a function $A:\Omega\to\mathbb{R}$ be of class $C^3(\Omega)$. Then from Taylor's expansion about (x_0,y_0) of the function of two variables, we get

$$\begin{aligned}A(x,y) &= A(x_0,y_0)+\frac{1}{1!}[A_x(x_0,y_0)(x-x_0)+A_y(x_0,y_0)(y-y_0)]+\\ &+ \frac{1}{2!}[A_{xx}(x_0,y_0)(x-x_0)^2+2A_{xy}(x_0,y_0)(x-x_0)(y-y_0)+A_{yy}(x_0,y_0)(y-y_0)^2]+\\ &+ \frac{1}{3!}[A_{xxx}(x_0,y_0)(x-x_0)^3+3A_{xxy}(x_0,y_0)(x-x_0)^2(y-y_0)+\\ &+ 3A_{xyy}(x_0,y_0)(x-x_0)(y-y_0)^2+A_{yyy}(x_0,y_0)(y-y_0)^3]++o(\|(x-x_0,y-y_0)\|^3)\,,\end{aligned}$$

where $\|(x-x_0,y-y_0)\|=\sqrt{(x-x_0)^2+(y-y_0)^2}$, and $o(h)$ is the function for which $\lim\limits_{h\to0}\frac{o(h)}{h}=0$.

For $x_0=y_0=\alpha$, $x=x_n$ and $y=\tilde{x}_n$ it is easy to obtain

$$\begin{aligned}&A(x_n,\tilde{x}_n)=A(\alpha,\alpha)+\frac{1}{1!}[A_x(\alpha,\alpha)(x_n-\alpha)+A_y(\alpha,\alpha)(\tilde{x}_n-\alpha)]+\\ &+\frac{1}{2!}[A_{xx}(\alpha,\alpha)(x_n-\alpha)^2+2A_{xy}(\alpha,\alpha)(x_n-\alpha)(\tilde{x}_n-\alpha)+A_{yy}(\alpha,\alpha)(\tilde{x}_n-\alpha)^2]+\\ &+\frac{1}{3!}[A_{xxx}(\alpha,\alpha)(x_n-\alpha)^3+3A_{xxy}(\alpha,\alpha)(x_n-\alpha)^2(\tilde{x}_n-\alpha)+\\ &+3A_{xyy}(\alpha,\alpha)(x_n-\alpha)(\tilde{x}_n-\alpha)^2+A_{yyy}(\alpha,\alpha)(\tilde{x}_n-\alpha)^3]+o(\|(x_n-\alpha,\tilde{x}_n-\alpha)\|^3)\,.\end{aligned}$$

Since $x_n-\alpha=e_n$, $\tilde{x}_n-\alpha=c_2e_n^2+2(c_3-c_2^2)e_n^3+o(e_n^3)$, we have that

$$\begin{aligned}o(\|(x_n-\alpha,\tilde{x}_n-\alpha)\|^3) &= o((\sqrt{(x_n-\alpha)^2+(\tilde{x}_n-\alpha)^2})^3)\\ &= o((\sqrt{e_n^2+(c_2e_n^2+2(c_3-c_2^2)e_n^3+o(e_n^3))^2})^3)=o(e_n^3).\end{aligned}$$

From the idempotency of A, we have

$$\begin{aligned}A(x_n,\tilde{x}_n)-\alpha &= e_nA_x(\alpha,\alpha)+e_n^2[A_y(\alpha,\alpha)c_2+\frac{1}{2}A_{xx}(\alpha,\alpha)]+e_n^3[2A_y(\alpha,\alpha)(c_3-c_2^2)+\\ &+ A_{xy}(\alpha,\alpha)c_2+\frac{1}{6}A_{xxx}(\alpha,\alpha)]+o(e_n^3)\,.\end{aligned}$$

Therefore, using Taylor's expansion of f' about α, we get

$f'(A(x_n,\tilde{x}_n))=f'(\alpha)+f''(\alpha)\big(A(x_n,\tilde{x}_n)-\alpha\big)+\frac{f'''(\alpha)}{2!}\big(A(x_n,\tilde{x}_n)-\alpha\big)^2+\frac{f^{IV}(\alpha)}{3!}\big(A(x_n,\tilde{x}_n)-\alpha\big)^3+o(\big(A(x_n,\tilde{x}_n)-\alpha\big)^3)$

$=f'(\alpha)+e_nf''(\alpha)A_x(\alpha,\alpha)+e_n^2\big[f''(\alpha)\big(A_y(\alpha,\alpha)c_2+\frac{1}{2}A_{xx}(\alpha,\alpha)\big)+\frac{f'''(\alpha)}{2}A_x^2(\alpha,\alpha)\big]+e_n^3\big[f''(\alpha)\big(2A_y(\alpha,\alpha)(c_3-c_2^2)+A_{xy}(\alpha,\alpha)c_2+\frac{1}{6}A_{xxx}(\alpha,\alpha)\big)+\frac{f'''(\alpha)}{2}\cdot2A_x(\alpha,\alpha)\big(A_y(\alpha,\alpha)c_2+\frac{1}{2}A_{xx}(\alpha,\alpha)\big)+\frac{f^{IV}(\alpha)}{6}A_x^3(\alpha,\alpha)\big]+o(e_n^3)$

$$= f'(\alpha)\Big[1+e_n 2c_2A_x(\alpha,\alpha)+e_n^2\big[c_2\big(2c_2A_y(\alpha,\alpha)+A_{xx}(\alpha,\alpha)\big)+3c_3A_x^2(\alpha,\alpha)\big]$$
$$+e_n^3\big[2c_2\big(2A_y(\alpha,\alpha)(c_3-c_2^2)+A_{xy}(\alpha,\alpha)c_2+\tfrac{1}{6}A_{xxx}(\alpha,\alpha)\big)$$
$$+6c_3A_x(\alpha,\alpha)\big(A_y(\alpha,\alpha)c_2+\tfrac{1}{2}A_{xx}(\alpha,\alpha)\big)+4c_4A_x^3(\alpha,\alpha)\big]+o(e_n^3)\Big] = f'(\alpha)(1+S).$$

Now, expression (3.1) can be transformed into

$$\begin{aligned}
x_{n+1} &= x_n - \tfrac{f(x_n)}{f'(A(x_n,\tilde{x}_n))}\\
&= x_n - \tfrac{f'(\alpha)(e_n+c_2e_n^2+c_3e_n^3+o(e_n^3))}{f'(\alpha)(1+S)}\\
&= x_n - \big(e_n+c_2e_n^2+c_3e_n^3+o(e_n^3)\big)(1+S)^{-1}\\
&= x_n - \big(e_n+c_2e_n^2+c_3e_n^3+o(e_n^3)\big)\big(1-S+S^2+o(S^2)\big)\\
&= x_n - \big(e_n+c_2e_n^2+c_3e_n^3+o(e_n^3)\big)\cdot\big[1+e_n\big(-2c_2A_x(\alpha,\alpha)\big)\\
&\quad +e_n^2\big(-2c_2^2A_y(\alpha,\alpha)-c_2A_{xx}(\alpha,\alpha)-3c_3A_x^2(\alpha,\alpha)+4c_2^2A_x^2(\alpha,\alpha)\big)+o(e_n^2)\big]\\
&= x_n - e_n + e_n^2\Big(2c_2A_x(\alpha,\alpha)-c_2\Big) - e_n^3\Big(-2c_2^2A_y(\alpha,\alpha)-c_2A_{xx}(\alpha,\alpha)\\
&\quad -3c_3A_x^2(\alpha,\alpha)+4c_2^2A_x(\alpha,\alpha)-2c_2^2A_x(\alpha,\alpha)+c_3\Big)+o(e_n^3)\\
&= \alpha + e_n^2c_2\Big(2A_x(\alpha,\alpha)-1\Big)+e_n^3\Big(c_2A_{xx}(\alpha,\alpha)+3c_3A_x^2(\alpha,\alpha)\\
&\quad +2c_2^2(A_y(\alpha,\alpha)-c_3-A_x(\alpha,\alpha))\Big)+o(e_n^3).
\end{aligned}$$

It yields

$$\begin{aligned}
e_{n+1} = &\ e_n^2c_2\Big(2A_x(\alpha,\alpha)-1\Big)\\
&+e_n^3\Big(c_2A_{xx}(\alpha,\alpha)+3c_3A_x^2(\alpha,\alpha)-c_3+2c_2^2(A_y(\alpha,\alpha)-A_x(\alpha,\alpha))\Big)+o(e_n^3).
\end{aligned}$$

On the other hand, A is also symmetric (i.e. $A_x(\alpha,\alpha)=1/2$), and the coefficient with e_n^2 vanishes. Therefore, the error equation has a form (3.2) which means that method IANM possesses convergence order three.

□

Let us recall some notions from functional analysis (see e.g. [15]).

If f is a mapping of the set of X to itself, then the point $x\in X$ is called a *fixed* (*stationary*) *point* of map f if $f(x)=x$.

We say that a map $f:X\to Y$ of the metric space (X,d_1) in the metric space (Y,d_2), is a *contraction* if there exists a real number $\lambda\in(0,1)$ such that for every $x_1,x_2\in X$

$$d_2(f(x_1),f(x_2))\le \lambda d_1(x_1,x_2).$$

The number λ is called the *coefficient of contraction*, and f a *contraction mapping*.

Theorem 2. (Banach fixed point theorem [15]) *If (X,d) is a complete metric space and $f:X\to X$ a contraction with coefficient λ, then there is one and only one fixed point $x\in X$ of the mapping f.*

Consider the function $\varphi(x) = x - \frac{f(x)}{g(x)}$, where $g(x) = f'\Big(A\big(x, x - \frac{f(x)}{f'(x)}\big)\Big)$. IANM can be rewritten shortly as

$$x_{n+1} = \varphi(x_n). \tag{3.7}$$

Theorem 3. *Let f be a real valued and $f \in C^3(I)$ on the closed interval I and α is a simple root of f belonging to the interior of I, then the function φ is a contraction in some neighbourhood of α in which the function value is assumed to be small enough.*

Proof. Clearly

$$\varphi'(x) = 1 - \frac{f'(x)}{g(x)} + \frac{f(x)g'(x)}{(g(x))^2}.$$

If we denote $\frac{\partial A}{\partial a}(a,b)$ as $A_a(a,b)$ and $\frac{\partial A}{\partial b}(a,b)$ as $A_b(a,b)$, then

$$\begin{aligned} g'(x) &= f''\Big(A\big(x, x - \frac{f(x)}{f'(x)}\big)\Big) \cdot \Big(A_a\big(x, x - \frac{f(x)}{f'(x)}\big) \cdot \frac{\partial a}{\partial x} + A_b\big(x, x - \frac{f(x)}{f'(x)}\big) \cdot \frac{\partial b}{\partial x}\Big) \\ &= f''\Big(A\big(x, x - \frac{f(x)}{f'(x)}\big)\Big) \cdot \Big(A_a\big(x, x - \frac{f(x)}{f'(x)}\big) + A_b\big(x, x - \frac{f(x)}{f'(x)}\big) \cdot \frac{f(x)f''(x)}{(f'(x))^2}\Big). \end{aligned}$$

Because $f(\alpha) = 0$ and the idempotency of A, it follows $g(\alpha) = f'(A(\alpha,\alpha)) = f'(\alpha)$ and by Lemma 1

$$g'(\alpha) = f''(A(\alpha,\alpha)) \cdot \Big(A_a(\alpha,\alpha) + A_b(\alpha,\alpha) \cdot 0\Big) = \frac{1}{2} f''(\alpha).$$

Using Taylor's expansion of $\varphi_1(x) = \frac{f'(x)}{g(x)}$ at point α, we obtain

$$\begin{aligned} \varphi_1(x) &= \varphi_1(\alpha) + \varphi_1'(\alpha)(x-\alpha) + \mathscr{O}((x-\alpha)^2) \\ &= \frac{f'(\alpha)}{g(\alpha)} + \frac{g'(\alpha)f'(\alpha) - g(\alpha)f''(\alpha)}{(f'(\alpha))^2}(x-\alpha) + \mathscr{O}((x-\alpha)^2) \\ &= 1 - \frac{f''(\alpha)}{2f'(\alpha)}(x-\alpha) + \mathscr{O}((x-\alpha)^2). \end{aligned} \tag{3.8}$$

Similarly, using Taylor's expansion of $\varphi_2(x) = \frac{f(x)g'(x)}{(g(x))^2}$ at point α, we obtain

$$\begin{aligned} \varphi_2(x) &= \varphi_2(\alpha) + \varphi_2'(\alpha)(x-\alpha) + \mathscr{O}((x-\alpha)^2) \\ &= \frac{f(\alpha)g'(\alpha)}{(g(\alpha))^2} + \frac{(f'(\alpha)g'(\alpha) + f(\alpha)g''(\alpha)) \cdot g^2(\alpha) - f(\alpha)g'(\alpha) \cdot 2g(\alpha)(g'(\alpha))^2}{g^4(\alpha)} \\ &\quad \cdot (x-\alpha) + \mathscr{O}((x-\alpha)^2) \\ &= \frac{f''(\alpha)}{2f'(\alpha)}(x-\alpha) + \mathscr{O}((x-\alpha)^2). \end{aligned} \tag{3.9}$$

Now,

$$\begin{aligned}
|\varphi'(x)| &= |1-\varphi_1(x)+\varphi_2(x)| \\
&= |1-1+\frac{f''(\alpha)}{2f'(\alpha)}+\mathscr{O}((x-\alpha)^2)+\frac{f''(\alpha)}{2f'(\alpha)}+\mathscr{O}((x-\alpha)^2)| \\
&= |\frac{f''(\alpha)}{f'(\alpha)}(x-\alpha)+\mathscr{O}((x-\alpha)^2)| \\
&\leq |\frac{f''(\alpha)}{f'(\alpha)}|\cdot|x-\alpha|+|\mathscr{O}((x-\alpha)^2)|. \qquad (3.10)
\end{aligned}$$

If we use the definition of the symbol $\mathscr{O}$, we assume that there exists a positive real number M and a positive real number δ such that for all x from the neighbourhood $(\alpha-\delta,\alpha+\delta)$ of point α

$$\mathscr{O}((x-\alpha)^2)\leq M|x-\alpha|^2$$

is satisfied.

If we introduce the label $L=\frac{f''(\alpha)}{f'(\alpha)}$, it follows

$$|\varphi'(x)| \leq (L+M\cdot|x-\alpha|)|x-\alpha|\leq(L+M\delta)|x-\alpha|.$$

If we choose the neighborhood $(\alpha-\varepsilon,\alpha+\varepsilon)$ of the point α such that

$$\varepsilon=\min\Big\{\delta,\frac{1}{2(L+M\delta)}\Big\}, \qquad (3.11)$$

we get that $|\varphi'(x)|<\frac{1}{2}$, i.e., φ is a contraction. □

Furthermore, for ε defined by (3.11) and arbitrary $x\in[\alpha-\varepsilon,\alpha+\varepsilon]=I$, we get

$$|\varphi(x)-\alpha|=|\varphi(x)-\varphi(\alpha)|=|\varphi'(\bar{x})|\cdot|x-\alpha|<|x-\alpha|<\varepsilon,$$

i.e., $\varphi(x)\in[\alpha-\varepsilon,\alpha+\varepsilon]$ and $\varphi:I\to I$.

Thus, conditions of the Banach fixed point theorem are fulfilled. Hence, the mapping φ has a unique fixed point in such a neighbourhood, and therefore the same follows for the function f, then IANM ensures the sequence $\{x_n\}$ defined by (3.7) converges to the value α for any initial iteration x_0 sufficiently close to α.

4 Numerical Results and Conclusion

In order to verify the theoretical results from the previous section, four test examples have been employed. The first three examples are derived from the real-life problems – Planck's radiation law, the electron trajectory in the air gap between two parallel plates, Van der Wall's equation of state (for more details see [8, 11]). The fourth example is a standard test problem taken from [16].

Test examples:

$f_1(x) = e^{-x} - 1 + x/5;\; x_0 = 3;\; \alpha = 4.9651...$
$f_2(x) = x - 0.5\cos x + \pi/4;\; x_0 = -1.4;\; \alpha = -0.3090...$
$f_3(x) = 0.986x^3 - 5.181x^2 + 9.067x - 5.289;\; x_0 = 2;\; \alpha = 1.9298...$
$f_4(x) = \left(1 - \sin x^2\right)\frac{x^2+1}{x^3+1} + x\log\left(x^2 - \pi + 1\right) - \frac{1+\pi}{1+\sqrt{\pi^3}};\; x_0 = 1.7;\; \alpha = \sqrt{\pi}.$

The numerical results are displayed in Tables 1-4. Along with the Newton's method in the first row, the tables present the numerical performance of families (3.1) and (1.5) for various aggregation functions. In fact, in order to compare the efficiency of the inner and the external aggregation Newton's method, each mean has been used to construct special cases of families (1.5) and (3.1). Hence, the inner aggregation Newton's methods are denoted by suffix "i", while their external counterparts are denoted by suffix "e". The numerical values are organized in four columns. '*it*' column represents the number of the iterations and '*nFe*' column relates to the number of function/derivative evaluations required to satisfy the stopping criterion $|x_{n+1} - x_n| + |f(x_{n+1})| < 10^{-7}$. Column $|x_3 - x_2|$ shows the absolute value of the two consecutive approximations difference. The last column displays the computational order of convergence COC (due to [17]) calculated by

$$\mathrm{COC} = \frac{\ln|(x_{n+1}-\alpha)/(x_n-\alpha)|}{\ln|(x_n-\alpha)/(x_{n-1}-\alpha)|}.$$

The Wolfram Mathematica programming package (ver. 11.0) has been used for performing all numerical computations.

Table 1: Numerical results for $f_1(x)$

method	it	nFe	$\lvert x_3 - x_2\rvert$	COC
N	4	8	0.001886	1.9504
Ai	4	12	$5.692\cdot 10^{-7}$	3.0040
Ae	4	12	$6.479\cdot 10^{-7}$	2.9970
Hi	3	9	$3.748\cdot 10^{-13}$	3.5817
He	4	12	$2.673\cdot 10^{-6}$	2.9944
Gi	3	9	$4.938\cdot 10^{-8}$	3.4863
Ge	4	12	$1.414\cdot 10^{-6}$	2.9958
Qi	4	12	$2.126\cdot 10^{-6}$	3.0060
Qe	4	12	$2.394\cdot 10^{-7}$	2.9981
Ci	4	12	$4.889\cdot 10^{-6}$	3.0076
Ce	3	9	$6.083\cdot 10^{-8}$	3.1845
CHi	4	12	$5.250\cdot 10^{-6}$	3.0078
CHe	3	9	$5.753\cdot 10^{-8}$	3.1791
Z_pi $(p = 1/4)$	4	12	$1.096\cdot 10^{-7}$	3.0024
Z_pe $(p = 1/4)$	4	12	$1.182\cdot 10^{-6}$	2.9961
H_pi $(p = 1)$	4	12	$3.033\cdot 10^{-7}$	3.0033

Table 1: Numerical results for $f_1(x)$

method	it	nFe	$\|x_3 - x_2\|$	COC
H_pe $(p=1)$	4	12	$8.569 \cdot 10^{-7}$	2.9966
Q_pi $(p=9)$	4	12	$6.012 \cdot 10^{-5}$	3.0158
Q_pe $(p=9)$	4	12	$4.287 \cdot 10^{-7}$	2.9999
C_pi $(p=3)$	4	12	$1.524 \cdot 10^{-5}$	3.0106
C_pe $(p=3)$	3	9	$3.118 \cdot 10^{-11}$	2.7593

Table 2: Numerical results for $f_2(x)$

method	it	nFe	$\|x_3 - x_2\|$	COC
N	5	10	$2.067 \cdot 10^{-2}$	1.9963
Ai	4	12	$1.069 \cdot 10^{-5}$	3.0071
Ae	4	12	$4.027 \cdot 10^{-4}$	3.0424
Hi	5	15	$2.338 \cdot 10^{-2}$	3.0003
He	4	12	$1.713 \cdot 10^{-6}$	3.0082
Gi	5	15	$9.398 \cdot 10^{-3}$	3.0002
Ge	4	12	$6.552 \cdot 10^{-5}$	3.0232
Qi	4	12	$2.575 \cdot 10^{-4}$	3.0150
Qe	4	12	$1.258 \cdot 10^{-3}$	3.0628
Ci	4	12	$2.444 \cdot 10^{-3}$	2.9995
Ce	4	12	$2.707 \cdot 10^{-3}$	3.0816
CHi	5	15	$1.600 \cdot 10^{-2}$	2.9995
CHe	4	12	$2.979 \cdot 10^{-3}$	3.0856
Z_pi $(p=1/4)$	4	12	$4.997 \cdot 10^{-3}$	3.0398
Z_pe $(p=1/4)$	4	12	$1.132 \cdot 10^{-4}$	3.0277
H_pi $(p=1)$	4	12	$5.249 \cdot 10^{-4}$	3.0178
H_pe $(p=1)$	4	12	$2.427 \cdot 10^{-4}$	3.0357
Q_pi $(p=9)$	5	15	$5.196 \cdot 10^{-2}$	3.0047
Q_pe $(p=9)$	5	15	$4.666 \cdot 10^{-2}$	3.0109
C_pi $(p=3)$	5	15	$2.002 \cdot 10^{-2}$	2.9995
C_pe $(p=3)$	5	15	$8.536 \cdot 10^{-3}$	3.0013

Table 3: Numerical results for $f_3(x)$

method	it	nFe	$\|x_3 - x_2\|$	COC
N	5	10	$1.825 \cdot 10^{-3}$	1.9917
Ai	4	12	$5.290 \cdot 10^{-6}$	2.9762
Ae	4	12	$1.022 \cdot 10^{-5}$	2.9690
Hi	4	12	$4.514 \cdot 10^{-6}$	2.9776

Table 3: Numerical results for $f_3(x)$

method	it	nFe	$\lvert x_3 - x_2 \rvert$	COC
He	3	9	$9.040 \cdot 10^{-8}$	2.7459
Gi	4	12	$4.891 \cdot 10^{-6}$	2.9769
Ge	4	12	$2.059 \cdot 10^{-6}$	2.9841
Qi	4	12	$5.711 \cdot 10^{-6}$	2.9755
Qe	4	12	$2.936 \cdot 10^{-5}$	2.9526
Ci	4	12	$6.156 \cdot 10^{-6}$	2.9748
Ce	4	12	$6.218 \cdot 10^{-5}$	2.9356
CHi	4	12	$6.156 \cdot 10^{-6}$	2.9748
CHe	4	12	$6.550 \cdot 10^{-5}$	2.9343
Z_pi $(p = 1/4)$	4	12	$4.989 \cdot 10^{-6}$	2.9767
Z_pe $(p = 1/4)$	4	12	$3.315 \cdot 10^{-6}$	2.9805
H_pi $(p = 1)$	4	12	$5.154 \cdot 10^{-6}$	2.9764
H_pe $(p = 1)$	4	12	$6.489 \cdot 10^{-6}$	2.9742
Q_pi $(p = 9)$	4	12	$9.359 \cdot 10^{-6}$	2.9704
Q_pe $(p = 9)$	5	15	$9.122 \cdot 10^{-4}$	2.9963
C_pi $(p = 3)$	4	12	$7.118 \cdot 10^{-6}$	2.9733
C_pe $(p = 3)$	4	12	$1.904 \cdot 10^{-4}$	2.8966

Table 4: Numerical results for $f_4(x)$

method	it	nFe	$\lvert x_3 - x_2 \rvert$	COC
N	4	8	$1.844 \cdot 10^{-5}$	2.0002
Ai	3	9	$5.470 \cdot 10^{-13}$	2.9913
Ae	3	9	$3.337 \cdot 10^{-10}$	3.0283
Hi	3	9	$4.741 \cdot 10^{-12}$	2.9843
He	3	9	$3.122 \cdot 10^{-11}$	3.0447
Gi	3	9	$1.860 \cdot 10^{-12}$	2.9870
Ge	3	9	$1.215 \cdot 10^{-10}$	3.0342
Qi	3	9	$9.309 \cdot 10^{-14}$	2.9994
Qe	3	9	$7.444 \cdot 10^{-10}$	3.0244
Ci	3	9	$3.544 \cdot 10^{-15}$	3.0253
Ce	3	9	$1.445 \cdot 10^{-9}$	3.0213
CHi	3	9	$3.535 \cdot 10^{-15}$	3.0251
CHe	3	9	$1.450 \cdot 10^{-9}$	3.0215
Z_pi $(p = 1/4)$	3	9	$1.416 \cdot 10^{-12}$	2.9879
Z_pe $(p = 1/4)$	3	9	$1.605 \cdot 10^{-10}$	3.0325
H_pi $(p = 1)$	3	9	$8.585 \cdot 10^{-13}$	2.9897
H_pe $(p = 1)$	3	9	$2.451 \cdot 10^{-10}$	3.0300
Q_pi $(p = 9)$	3	9	$6.411 \cdot 10^{-12}$	2.9660
Q_pe $(p = 9)$	4	12	$1.950 \cdot 10^{-8}$	3.0000
C_pi $(p = 3)$	3	9	$1.780 \cdot 10^{-14}$	2.9525
C_pe $(p = 3)$	3	9	$4.191 \cdot 10^{-9}$	3.0170

According to the all calculated COC values (all those values are very close to three), it is obvious that the theoretical results related to the convergence order agree with the numerical results. Note that in some cases, for the same test example, the inner aggregation variant of the method is more efficient than its external counterpart (see, for example Hi and He, Gi and Ge for f_1), but for some other aggregation function the opposite behavior can be detected (see, for example Q_pi and Q_pe, CHi and CHe for f_1). In general, the efficiency of the methods depends not only on the choice of aggregation function, but also on the initial approximation and, naturally, on the nonlinear equation itself. Therefore, it cannot be claimed that the inner aggregation family is better than the external aggregation methods, and *vice versa*. Nevertheless, it is very clear that family (3.1) can be a valuable alternative to the existing third-order methods.

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