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The Principal Extrinsic and Intrinsic Tangent Directions of Generalised Wintgen Ideal Legendrian Submanifolds

A. Šebeković

Abstract: For Legendrian submanifolds $\mathbb{M}^n$ in Sasakian space forms $\tilde{\mathbb{M}}^{2n+1}(c)$, I. Mihai obtained an inequality relating (intrinsic) the normalised scalar curvature and (extrinsic) squared mean curvature and normalised normal curvature of $\mathbb{M}$ in $\tilde{\mathbb{M}}$, characterising also the corresponding equality case. In this paper, it's shown that (intrinsic) Ricci principal directions and (extrinsic) Casorati principal directions, for generalised Wintgen ideal Legendrian submanifolds $\mathbb{M}^n$ in Sasakian space forms $\tilde{\mathbb{M}}^{2n+1}(c)$, do coincide.

Keywords: Generalised Wintgen ideal Legendrian submanifolds, Ricci principal directions, Casorati principal directions

1 Preliminaries

The main and the most naturale Riemannian invariants are the curvature invariants: sectional, scalar, Ricci curvatures...

For surfaces $\mathbb{M}^2$ in $\mathbb{E}^3$, the Euler inequality $K \leq H^2$, where K is Gauss curvature of $\mathbb{M}^2$ (intrinsic) and H^2 is squared mean curvature of $\mathbb{M}^2$ in $\mathbb{E}^3$ (extrinsic), and equality in this case hold if and only if $\mathbb{M}^2$ is totally umbilical in $\mathbb{E}^3$ or still, by a theorem of Meusnier, if and only if $\mathbb{M}^2$ is a part of a plane $\mathbb{E}^2$ or a round spfere $\mathbb{S}^2$ in $\mathbb{E}^3$. For surfaces $\mathbb{M}^2$ in $\mathbb{E}^4$, Wintgen [17] (1979) proved that Gauss curvature K and squared mean curvature H^2 and normal curvature $K^{\perp}$ of $\mathbb{M}^2$ satisfy the inquality $K \leq H^2 - K^{\perp}$. The equality in this case holds if and only if the curvature elipses $\varepsilon = \{h(u,u) | u \in T\mathbb{M} \text{ and } ||u|| = 1\}$ in the normal planes of $\mathbb{M}^2$ in $\mathbb{E}^4$ are circles.

This Wintgen inequality between the most important intrinsic and extrinsic scalar valued curvatures of surfaces $\mathbb{M}^2$ in $\mathbb{E}^4$ was shown to hold more generally for all surfaces $\mathbb{M}^2$ in arbitrary dimensional space forms $\tilde{\mathbb{M}}^{2+m}(c)$, inclusive the above characterisation of

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A. Šebeković is with the State University of Novi Pazar, Novi Pazar, Serbia

the equality case by Rouxel [15] and by Guadalupe and Rodriguez [9]. De Smet, Dillen, Verstraelen and Vrancken [7] in 1999. proved the generalised Wintgen inequaluty

$$\rho \leq H^2 - \rho^{\perp} + c \tag{1}$$

for all n-dimensional submanifolds $\mathbb{M}^n$ with co-dimension $m = 2$ in real space forms $\tilde{\mathbb{M}}^{n+2}(c)$; ρ is the normalised scalar curvature of $\mathbb{M}^n$ defined by $\rho = \frac{2}{n(n-1)} \sum_{i<j}^{n} < R(e_i, e_j)e_j, e_i >$, and $\rho^{\perp}$ is the normalised normal scalar curvature function of $\mathbb{M}^n$ at a point p, defined by

$$\rho^{\perp}(p) = \frac{2}{n(n-1)} \sqrt{\sum_{i<j}^{n} \sum_{r<s}^{2} < R^{\perp}(e_i, e_j)\xi_r, \xi_s >^2},$$

where by $\{e_1, \ldots, e_n\}$ is any orthonormal basis of the $T_p\mathbb{M}^n$ $(p \in \mathbb{M}^n)$, R is Riemann-Christoffel curvature tensor of $\mathbb{M}^n$ and $R^{\perp}$ is the curvature tensor of normal space and $\{\xi_1, \xi_2\}$ is an orthonormal basis of the normal space. They also characterised the equality case in terms of the shape operators of $\mathbb{M}^n$ in $\tilde{\mathbb{M}}^{n+2}(c)$ and also conjectured (1) to hold for all n-dimensional submanifolds $\mathbb{M}^n$ in real space forms $\tilde{\mathbb{M}}^{n+m}(c)$ of arbitrary co-dimension.

Choi an Lu [5], Lu [12] and Ge-Thang [8] proved this conjecture and also gave a characterisation of the equality case in terms of the second fundimental form.

The submanifolds $\mathbb{M}^n$ in $\tilde{\mathbb{M}}^{n+m}(c)$ satisfying equality in Wintgen inequality (1) are called ***Wintgen ideal submanifolds***. For many examples and geometrical properties of such submanifolds, see e.g [3, 5, 7, 8, 9, 10, 12, 14, 15].

The next step in generalisation of Wintgen ideal submanifolds is given bay I. Mihai [13].

2 Generalised Wintgen ideal Legendrian submanifolds of Sasakian space forms

A $(2m+1)$-dimensional Riemannian manifolds $(\tilde{\mathbb{M}}^{2m+1}, g)$ is Sasakian manifolds if it admits an endomorphisam ϕ of its tangent bundle $T\tilde{\mathbb{M}}^{2m+1}(c)$, a vector field ξ and a $1-$ form η satisfying

$$\phi^2 = -id + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0,$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi),$$

$$(\tilde{\nabla}_X \phi)Y = -g(X,Y)\xi + \eta(Y)X, \quad \tilde{\nabla}_X \xi = \phi X,$$

for vector fields X, Y on $\tilde{\mathbb{M}}^{2m+1}$. With $\tilde{\nabla}$ is denote Riemannian connection with respect to g. A plane section π in $T_p\tilde{\mathbb{M}}^{2m+1}$ is called ϕ-section if it is spanned by X and ϕX, where X is unit tangent vector ortogonal to characteristic vector filed ξ. The sectional curvature of ϕ-

section is called a ϕ-sectional curvature and a Sasakian manifolds with constant ϕ-sectional curvature c is said to be a Sasakian space form and is denoted by $\overset{\sim}{\mathbb{M}}^{2m+1}(c)$. The curvature tensor of Sasakian space forms $\overset{\sim}{\mathbb{M}}^{2m+1}(c)$ is given by [1]

$$\begin{aligned}
\widetilde{R}(X,Y)Z &= \frac{c+3}{4}\{g(Y,Z)X - g(X,Z)Y\} + \\
&+ \frac{c-1}{4}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - \\
&- g(Y,Z)\eta(X)\xi + g(\phi Y,Z)\phi X - \\
&- g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z\},
\end{aligned}$$

where X, Y, Z are any tangent vector fields on $\overset{\sim}{\mathbb{M}}^{2m+1}(c)$. If $\mathbb{M}^n$ is n- dimensional submanifolds in a Sasakian space form $\overset{\sim}{\mathbb{M}}^{2m+1}(c)$ then the Gauss equation is given by

$$\widetilde{R}(X,Y,Z,W) = R(X,Y,Z,W) + g(h(X,W),h(Y,Z)) - g(h(X,Z),h(Y,W)),$$

where by R and h are the Riemann curvature tensor and second fundamental form, respectively, of $\mathbb{M}^n$, and X, Y, Z, W are vector tangent to $\mathbb{M}^n$. The mean curvature vector is given by $H(p) = \frac{1}{n}\sum_{i=1}^{n} h(e_i,e_i)$, where $\{e_1,\ldots,e_n,e_{n+1},\ldots,e_{2m+1}\}$ is an orthonormal basis of tangent space $\overset{\sim}{\mathbb{M}}^{2m+1}(c)$, such that $\{e_1,e_2,\ldots,e_n\}$ are tangent to $\mathbb{M}^n$ at p. A submanifold $\mathbb{M}^n$ normal to ξ in a Sasakian manifold is said to be a C- totally real submanifold. It follows that $\phi(T_p\mathbb{M}^n) \subset T_p^{\perp}\mathbb{M}^n$, for every p in C-totally real submanifold $\mathbb{M}^n$. If $m = n$, then $\mathbb{M}^n$ is called ***Legendrain submanifold***

Let $\mathbb{M}^n$ be n-dimensional Legendrian submanifold of a Sasakian space form $\overset{\sim}{\mathbb{M}}^{2m+1}(c)$ and $\{e_1,e_2,\ldots,e_n\}$ an orthonormal frame on $\mathbb{M}^n$ and $\{e_{n+1},\ldots,e_{2n},e_{2n+1} = \xi\}$ an orthonormal frame in the normal bundle $T^{\perp}\mathbb{M}^n$.

Then the Gauss equation is given by:

$$\begin{aligned}
R(X,Y,Z,W) &= \frac{c+3}{4}\{g(X,Z)g(Y,W) - g(Y,Z)g(X,W)\} \\
&+ g(h(X,Z)h(Y,W)) - g(h(X,W),h(Y,Z)),
\end{aligned}$$

where h and A denote the second fundamental form and the shape operator of $\mathbb{M}^n$ in $\overset{\sim}{\mathbb{M}}^{2m+1}(c)$.

Theorem 2.1 *[13] Let $\mathbb{M}^n$ be an n-dimensional Legendrian submanifold of a Sasakian space form $\overset{\sim}{\mathbb{M}}^{2m+1}(c)$. Then*

$$(\rho^{\perp})^2 \leq (||H||^2 - \rho + \frac{c+3}{4})^2 + \frac{4}{n(n-1)}(\rho - \frac{c+3}{4})\frac{c-1}{4} + \frac{(c-1)^2}{8n(n-1)} \qquad (2)$$

and equality hold if and only if with respect to suitable orthonormal frames $\{e_1,\ldots,e_n\}$ *and* $\{e_{n+1},\ldots,e_{2n},e_{2n+1}=\xi\}$, *the shape operators of* $\mathbb{M}^n$ *in* $\overset{\sim}{\mathbb{M}}^{2m+1}(c)$ *are given by:*

$$A_{e_{n+1}}=\begin{bmatrix}\lambda_1 & \mu & 0 & \cdots & 0\\ \mu & \lambda_1 & 0 & \cdots & 0\\ 0 & 0 & \lambda_1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & \lambda_1\end{bmatrix},\quad A_{e_{n+2}}=\begin{bmatrix}\lambda_2+\mu & 0 & 0 & \cdots & 0\\ 0 & \lambda_2-\mu & 0 & \cdots & 0\\ 0 & 0 & \lambda_2 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & \lambda_2\end{bmatrix},$$

$$A_{e_{n+3}}=\begin{bmatrix}\lambda_3 & 0 & 0 & \cdots & 0\\ 0 & \lambda_3 & 0 & \cdots & 0\\ 0 & 0 & \lambda_3 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & \lambda_3\end{bmatrix},\quad A_{e_{n+4}}=\cdots=A_{e_{2n}}=A_{e_{2n+1}}=0,$$

where by λ_1, λ_2, λ_3 *and* μ *are real functions on* $\mathbb{M}^n$.

Legendrian submanifolds $\mathbb{M}^n$ in Sasakian space forms $\overset{\sim}{\mathbb{M}}^{2m+1}(c)$ satisfying equality in generalised Wintgen inequality (2) are called ***generalised Wintgen ideal Legendrian sumanifolds***. A frame $\{e_1,e_2,\ldots,e_n,e_{n+1},\ldots,e_{2n},e_{2n+1}\}$ with the correspoding shape operators from Theorem 2.1 is called a Choi-Lu frame on such $\mathbb{M}^n$ in $\overset{\sim}{\mathbb{M}}^{2m+1}(c)$ and its distinguished tangent plane $e_1\wedge e_2$ is called the Choi-Lu plane of generalised Wintgen ideal Legendrian submanifolds concerned.

3 The Casorati principal directions of submanifolds

For general submanifolds $\mathbb{M}^n$ in arbitrary Riemannian spaces $\overset{\sim}{\mathbb{M}}^{n+m}$, $(1,1)$ tensor field $A^C=\sum\limits_{\alpha}A_\alpha^2$, which is independent of the choice of local orthonormal normal frame fields $\{\xi_1,\ldots,\xi_m\}$ is called Casorati operator of $\mathbb{M}^n$ in $\overset{\sim}{\mathbb{M}}^{n+m}$. The Casorati curvature $C:\mathbb{M}^n\to\mathbb{R}$ of $\mathbb{M}^n$ in $\overset{\sim}{\mathbb{M}}^{n+m}$ is defined by $C=\frac{1}{n}trA^C=\frac{1}{n}\sum\limits_{\alpha,i,j}(h_{ij}^\alpha)^2$ whereby h_{ij}^α denote the components of the second fundamental form h with respect to any orthonormal frame field $\{e_1,\ldots,e_n,\xi_1,\ldots,\xi_m\}$ on $\mathbb{M}^n$ in $\overset{\sim}{\mathbb{M}}^{n+m}$. Since for each normal vector field ξ on $\mathbb{M}^n$ in $\overset{\sim}{\mathbb{M}}^{n+m}$ the shape operator A_ξ is a symmetric $(1,1)$ tensor field on $\mathbb{M}^n$ at every point $p\in\mathbb{M}^n$ all eigenvalues of $A_\xi(p)$ are real. Because of that, there exists on $\mathbb{M}^n$ an orthonormal set of eigenvector fields $F_1,\ldots,F_n$. By this set of vector fields is determined the (extrinsic) Casorati principal directions of $\mathbb{M}^n$ in $\overset{\sim}{\mathbb{M}}^{n+m}$. Corresponding eigenfunctions $c_1,\ldots,c_n$ (all ≥ 0)

are its extrinsic principal curvatures of $\mathbb{M}^n$ in $\overset{\sim}{\mathbb{M}}^{n+m}$, i.e. $A^C(F_i) = c_i F_i$. The geometrical meanings of these are given in [2, 11]

A hypersurface $\mathbb{M}^n$ in a Riemannian space $\overset{\sim}{\mathbb{M}}^{n+1}$ is called umbilical when its shape operator has eigenvalue of multiplicity n.

A hypersurfaces $\mathbb{M}^n$ in $\overset{\sim}{\mathbb{M}}^{n+1}$ is called quasi-umbilical and $2-$ quasi-umbilical when its shape operator has respectively eigen value of multiplicity $\geq n-1$ and $\geq n-2$. In the same way, submanifold $\mathbb{M}^n$ in some ambient Riemannian manifold $\overset{\sim}{\mathbb{M}}^{n+m}$ is called Casorati quasi-umbilical and Casorati $2-$ quasi-umbilical submanifolds $\mathbb{M}^n$ in $\overset{\sim}{\mathbb{M}}^{n+m}$.

From Theorem 2.1 it follows that Casorati operator of generalised Wintgen ideal Legendrian submanifolds $\mathbb{M}^n$ of Sasakian space form $\overset{\sim}{\mathbb{M}}^{2n+1}(c)$ is given by

$$A^C = \begin{bmatrix} \lambda + 2\lambda_2\mu + 2\mu^2 & 2\lambda_1\mu & 0 & \cdots & 0 \\ 2\lambda_1\mu & \lambda - 2\lambda_2\mu + 2\mu^2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda \end{bmatrix}$$

whereby $\lambda = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$. Its eigenvalues are:

$$c_1 = \lambda + 2\mu^2 + 2\mu\sqrt{\lambda_1^2 + \lambda_2^2},$$

$$c_2 = \lambda + 2\mu^2 - 2\mu\sqrt{\lambda_1^2 + \lambda_2^2},$$

$$c_3 = c_4 = \cdots = c_n = \lambda,$$

and corresponding eigenvectors are

$$F_1 = \frac{\lambda_1}{\sqrt{2}\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_2\sqrt{\lambda_1^2 + \lambda_2^2}}} e_1 - \frac{\lambda_2 + \sqrt{\lambda_1^2 + \lambda_2^2}}{\sqrt{2}\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_2\sqrt{\lambda_1^2 + \lambda_2^2}}} e_2,$$

$$F_2 = \frac{\lambda_1}{\sqrt{2}\sqrt{\lambda_1^2 + \lambda_2^2 - \lambda_2\sqrt{\lambda_1^2 + \lambda_2^2}}} e_1 - \frac{\lambda_2 - \sqrt{\lambda_1^2 + \lambda_2^2}}{\sqrt{2}\sqrt{\lambda_1^2 + \lambda_2^2 - \lambda_2\sqrt{\lambda_1^2 + \lambda_2^2}}} e_2,$$

$$F_k = e_k, \ (k = 3, 4, \ldots, n).$$

The vector fields F_1, F_2 determine the 1-dimensional eigenspaces of A^C corresponding to c_1 and c_2 respectively, unless when $\lambda_1 = \lambda_2 = 0$ and $\mu \neq 0$, in which case the Choi-Lu plane itself is 2-dimensional eigenspace of A^C. When $\mu = 0$ Casorati principal directions are undetermined, and A^C is proportional to the identity operator ($\mathbb{M}^n$ is totaly umbilical). In any case, the tangent subspace $e_3 \wedge \ldots \wedge e_n$ of $\mathbb{M}^n$ is an $(n-2)$-dimensional eigenspace of A^C corresponding to the Casorati curvature λ. Hence, in particular we have the following.

Theorem 3.1 *Every generalised Wintgen ideal Legendrian submanifold* $\mathbb{M}^n$ *in Sasakian space form* $\overset{\sim}{\mathbb{M}}^{2n+1}(c)$ *is Casorati* 2*-quasi-umbilical. When* $\mathbb{M}^n$ *is not totally umbilical, then the orthogonal complement of its Choi-Lu plane is its* $(n-2)$*-dimensional Casorati eigenspace.*

4 The Ricci principal directions of generalised Winthen ideal Legendrian submanifolds

From the Theorem 2.1 and Gauss equation we obtain, up to the Algebric symmetries of the $(0,4)$ curvature tensor R of the generalised Wintgen ideal Legedrian submanifold $\mathbb{M}^n$ of Sasakian space form $\overset{\sim}{\mathbb{M}}^{2n+1}(c)$, all components of R are zero except these:

$$\begin{aligned}
R_{1221} &= 2\mu^2 - c_1,\\
R_{1kk1} &= -\lambda_2\mu - c_1, \quad (k \geq 3)\\
R_{2kk2} &= \lambda_2\mu - c_1, \quad (k \geq 3)\\
R_{1kk2} &= -\lambda_1\mu, \quad (k \geq 3)\\
R_{kllk} &= -c_1, \quad (k \neq l,\ k,l \geq 3)
\end{aligned}$$

whereby $c_1 = \frac{c+3}{4} + \lambda_1^2 + \lambda_2^2 + \lambda_3^2$.

The nontrivial components of $(0,2)$ Ricci tensor S of generalised Wintgen ideal Legendrian submanifold $\mathbb{M}^n$ in Sasakian space form $\overset{\sim}{\mathbb{M}}^{2n+1}(c)$ are:

$$\begin{aligned}
S_{11} &= 2\mu^2 - (n-1)c_1 - (n-2)\lambda_2\mu\\
S_{22} &= 2\mu^2 - (n-1)c_1 + (n-2)\lambda_2\mu\\
S_{12} &= -(n-2)\lambda_1\mu\\
S_{kk} &= -(n-1)c_1, \quad (k \geq 3).
\end{aligned}$$

It follows that Ricci operator of such submanifold is given by

$$S = \begin{bmatrix}
2\mu^2-(n-1)c_1-(n-2)\lambda_2\mu & -(n-2)\lambda_1\mu & 0 & \cdots & 0\\
-(n-2)\lambda_1\mu & 2\mu^2-(n-1)c_1+(n-2)\lambda_2\mu & 0 & \cdots & 0\\
0 & 0 & -(n-1)c_1 & \cdots & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots\\
0 & 0 & 0 & \cdots & -(n-1)c_1
\end{bmatrix}$$

Its eigenvalues are

$$R_{ic_1} = 2\mu^2 - (n-1)c_1 + (n-2)\mu\sqrt{\lambda_1^2+\lambda_2^2},$$
$$R_{ic_2} = 2\mu^2 - (n-1)c_1 - (n-2)\mu\sqrt{\lambda_1^2+\lambda_2^2},$$
$$R_{ic_3} = R_{ic_4} = \cdots = R_{ic_n} = -(n-1)c_1$$

and corresponding eigenvector fields are:

$$R_1 = \frac{\lambda_1}{\sqrt{2}\sqrt{\lambda_1^2+\lambda_2^2+\lambda_2\sqrt{\lambda_1^2+\lambda_2^2}}}e_1 - \frac{\lambda_2+\sqrt{\lambda_1^2+\lambda_2^2}}{\sqrt{2}\sqrt{\lambda_1^2+\lambda_2^2+\lambda_2\sqrt{\lambda_1^2+\lambda_2^2}}}e_2,$$

$$R_2 = \frac{\lambda_1}{\sqrt{2}\sqrt{\lambda_1^2+\lambda_2^2-\lambda_2\sqrt{\lambda_1^2+\lambda_2^2}}}e_1 - \frac{\lambda_2-\sqrt{\lambda_1^2+\lambda_2^2}}{\sqrt{2}\sqrt{\lambda_1^2+\lambda_2^2-\lambda_2\sqrt{\lambda_1^2+\lambda_2^2}}}e_2,$$

$$R_k = e_k, \ \ (k = 3,4,\ldots,n).$$

Hence, in partikular we have the following.

Theorem 4.1 *Every generalised Wintgen ideal Legendrian submanifold $\mathbb{M}^n$ in Sasakian space form $\overset{\sim}{\mathbb{M}}^{2n+1}(c)$ is Ricci 2-quasi-umbilical. When $\mathbb{M}^n$ is not totally umbilical, then orthogonal complement of its Choi-Lu plane is its $(n-2)$-dimensional Ricci eigenspace*

The Casorati principal directions of a submanifold $\mathbb{M}^n$ in Riemannian space $\overset{\sim}{\mathbb{M}}^{n+m}$ are, from the extrinsic geometric point of view, the most important tangent directions. From the intrinsic geometric point of view, the Ricci principal directions of such submanifolds are the most important tangent directions.

The geometrical meaning of these notions could be seen in [6] where authors showed that for Wintgen ideal submanifolds in real space forms the Ricci principal directions coincide.

Here, from the corresponding fromulae given in sections 3 and 4, we establish the following.

Theorem 4.2 *On every generalised Wintgen ideal Legendrian submanifold $\mathbb{M}^n$ in Sasakian space form $\overset{\sim}{\mathbb{M}}^{2n+1}(c)$ the Casorati and Ricci principal directions do coincide*

Because of that, we may conclude that the particular shape any generalised Wintgen ideal Legendrian submanifold $\mathbb{M}^n$ does relise in ambient Sasakian space form $\overset{\sim}{\mathbb{M}}^{2n+1}(c)$ in order to undergo the very least possible amount of extrinsic stress as allowed by its normalise intrinsic Riemannian scalar curvature, manifests the geometrical property that the principal tangent directions which are determined by this shape, naimely its Casorati principal directions, are the same as the principal intrinsic tangent directions of its Riemannian structure, namely its Ricci principal directions.

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