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On Chua Dynamical System

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Abstract: The famous Chua piecewise dynamical system (Chua circuit) has been slightly modified by introducing smoothed h-function. So, the C^1 function is obtained instead of C^0one. Dynamics of such system is evidenced by numerical examination of the orbits in 3D phase space, Fourier spectrum, leading Lyapunov exponent bifurcation diagrams and Poincaré maps.

Keywords: Chua dynamics, chaos, strange attractors, bifurcation

1 Introduction

According to available literature, the first chaotic emergence of a three-dimensional flow was observed at Kyoto University by Yoshisuke Ueda in 1961. As latter being revealed, the dynamical system being observed was forced Duffing-van der Pol oscillator. Many other systems were also observed as chaotic although no reliable proofs have been offered. The first chaotic system that was observed in the laboratory, confirmed by computer simulation and rigorously mathematically proven was Chua's "double scroll" dynamical system ([3], [14], [15]). Originally created as an electric circuit, the Chua system (also called *Chua circuit*) has the following three dimensional representation

$$\begin{aligned} \frac{dx}{dt} &= \alpha\,(y-h(x))\,, \\ \frac{dy}{dt} &= x-y+z, \\ \frac{dz}{dt} &= -\beta y, \end{aligned} \tag{1}$$

where, h is the piecewise-linear function

$$h(x)=\begin{cases} m_1(x+1)-m_0, & x<-1, \\ m_0 x, & -1\le x\le 1, \\ m_1(x-1)+m_0, & x>1, \end{cases} \tag{2}$$

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and α and β are bifurcation parameters. The usual settings for the chaotic flow in the system is $\alpha = 8.8$, $\beta = 15$, $m_0 = -1/7$, $m_1 = 2/7$.

There is a slightly different model of Chua system

$$\begin{aligned} \frac{dx}{dt} &= \alpha\left(y - x - h(x)\right), \\ \frac{dy}{dt} &= x - y + z, \\ \frac{dz}{dt} &= -\beta y - \gamma z, \end{aligned} \tag{3}$$

with the same nonlinear function h, given by (2) and with an extra parameter γ. Typical values for slopes now are $m_0 = -8/7$, and $m_1 = -5/7$, while α and β may vary considerably.

After the first paper about the new dynamical system, called *Chua circuit* due the prototype was made in the form of electronic circuit, was published (1984), the number of publications steadily grows. Seven years after, the number of publications was roughly about 60. Ten years after this number increases to about 100. After twenty years (2004) the list of publications numbers 767 units. Such an interest is approved by high applicability of Chua system in different domains such as pseudo-random generators, cryptography, musical studies, biology etc. The system has capability of generating a surprisingly large number of topologically distinct chaotic attractors, as it was shown in nicely illustrated book [2] where near *thousand* attractors of Chua system was reported.

Many of these rich properties are consequence of the special geometry of the function h . For ex., it satisfies the symmetry property $h(-x) = -h(x)$, having m_0 as the middle slope and m_1 as side slopes (Fig. 1, left). Futher, this function has two break points $x = \pm 1$ with continuity C^0. Some time, it was an interesting issue to examine if these breakpoints are essential for generating chaotic dynamics in the system or not. What will happen if the original function h be smoothed to C^1 continuity? Will the chaos persist? In [17], Leonid Shilnikov has given an affirmative answer.

In this note, a special kind of smoothing of h is proposed and related systems are investigated on chaotic dynamics.

2 Smoothing h-function

The idea of replacing the piecewise linear h-function (2) by a smooth counterpart is not new. The first choice was a cubic polynomial, like $x \mapsto x^3/16 - x/6$ (see[1], [6], [7], [16], [19]). Tang and Man [18] have used piecewise quadratic function of the type $x \mapsto x(a + b|x|)$ with real parameters $a < 0$ and $b > 0$. Nice review of other different choices that include sinus function, hyperbolic functions or piecewise linear function with many breakpoints can be found in [2]. Such functions did not change qualitative behavior of chaotic dynamics of the system. Finally, Shilnikov [17] came with explanation that characteristic function h can be "any nonlinear function" of similar characteristics like cubic or "sigmoid" function $x \mapsto \dfrac{\exp(cx) - 1}{\exp(cx) + 1}$.

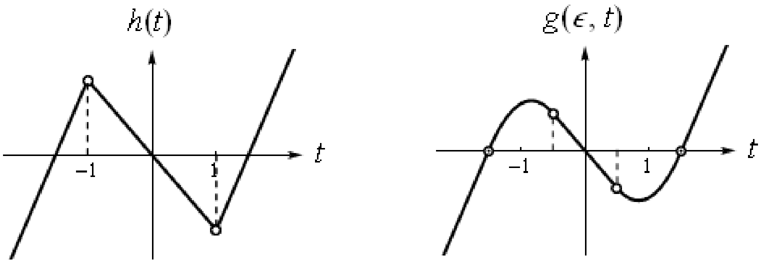

Fig. 1. Chua's h-function (left) and its C^1 approximation (right)

Our idea uses partial smoothing or replacement piecewise linear C^0 function by C^1 spline function that contains both linear and cubic parts. Such method was developed and used by these authors in some recent papers ([4], [5], [8], [9], [10]). The splines sometimes were produced as limit objects in subdivision procedures like Chaikin [10], (rational) de Casteljau [4] or Dyn-Levin-Gregory [5]. The opposite idea of approximation of smooth function by less smooth ones was also elaborated in [11], [12], [13]. Both processes have justification. Increasing of smoothness introduce more sensitive system parameters that provide finer tuning of the system. In addition, such parameters allow having better insight into the nature of chaos, its genesis and development. On the other hand, decreasing smoothness makes the corresponding system more suitable to apply in practice, since the function with piecewise transition characteristics is easier to model using electronic or mechanical components than the smooth ones. In fact, the chaotic dynamical system described by (1), (2) or (3) is dynamics of an electronic circuit invented by Leon Chua. The main component of Chua's circuit is a nonlinear resistor having electric characteristic described by function h. (sometimes called *Chua's diode*).

There are many different methods to get the function h smoothed. One of the simplest is to replace the restriction of h over the neighborhood of the points $x = -1$ and $x = 1$ with some simple function, say by polynomials. The simplest neighborhoods are spherical ε-neighborhoods, which means that these new functions must be defined over the intervals $[-1-\varepsilon,\ -1+\varepsilon]$, and $[1-\varepsilon,\ 1+\varepsilon]$. In order to continue h over these intervals with smoothness $\mathbf{C}^1$, the new functions must satisfy four end conditions, two positions $h(-1-\varepsilon),\ h(-1+\varepsilon)$ and two derivatives $h'(-1-\varepsilon),\ h'(-1+\varepsilon)$. Thus, the simplest polynomial functions meeting these Hermite type interpolating conditions are cubic polynomials, expressed in the most natural, Hermite cubic basis

$$\begin{aligned}\varphi_0(\tau) &= (1-\tau)^2(1+2\tau), \qquad \varphi_1(\tau) = \tau(1-\tau)^2,\\ \varphi_2(\tau) &= -\tau^2(1-\tau), \qquad\qquad \varphi_3(\tau) = \tau^2(3-2\tau).\end{aligned}$$

Taking into account interpolating conditions, two cubic polynomials can be constructed over this basis

$$p_{-1}(t) = h(-1-\varepsilon)\varphi_0(\tau) + h'(-1-\varepsilon)\varphi_1(\tau) + h'(-1+\varepsilon)\varphi_2(\tau) + h(-1+\varepsilon)\varphi_3(\tau),$$

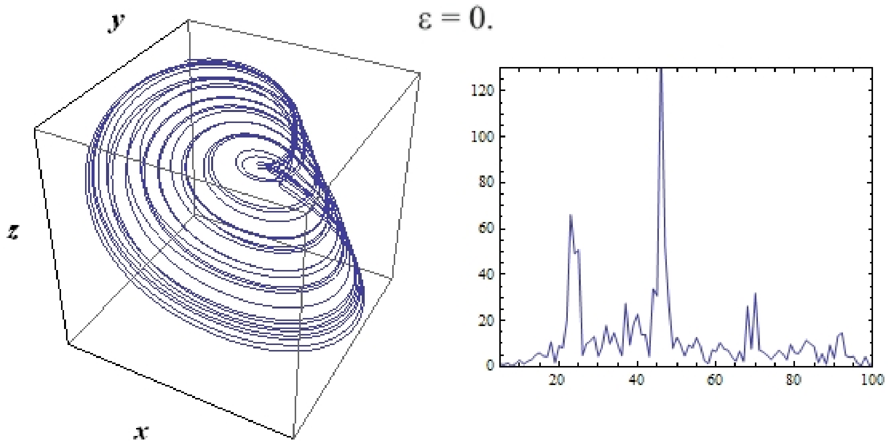

Fig. 2. Attractor of dynamical system (6) with $\varepsilon = 0.$, and the corresponding Fourier spectrum

where $\tau = \dfrac{t+1+\varepsilon}{2\varepsilon}$, for the left subinterval $t \in [-1-\varepsilon,\ -1+\varepsilon]$, and

$$p_1(t) = h(1-\varepsilon)\varphi_0(\tau) + h'(1-\varepsilon)\varphi_1(\tau) + h'(1+\varepsilon)\varphi_2(\tau) + h(1+\varepsilon)\varphi_3(\tau)$$

where $\tau = \dfrac{t-1+\varepsilon}{2\varepsilon}$, for the right subinterval$t \in [1-\varepsilon,\ 1+\varepsilon]$.

So, the new, C^1 function that approximates h is defined for all $0 \le \varepsilon \le 1$ by

$$g(\varepsilon,x) = \begin{cases} h(x), & x \in (-\infty,-1-\varepsilon) \cup [-1+\varepsilon, 1-\varepsilon) \cup [1+\varepsilon,+\infty) \\ p_{-1}(x), & x \in [-1-\varepsilon,-1+\varepsilon) \\ p_1(x), & x \in [1-\varepsilon, 1+\varepsilon) \end{cases} \tag{4}$$

The graph of the function $g(\varepsilon,\cdot)$ is given in Figure 1, right. It is easy to verify that $\dfrac{dg(\varepsilon,\, x)}{dx}$ is continuous over the real line. The modified Chua's dynamical system that we propose for further investigation is thou

$$\begin{aligned} \frac{dx}{dt} &= \alpha\,(y - x - g(\varepsilon,\, x))\,, \\ \frac{dy}{dt} &= x - y + z, \qquad\qquad 0 \le \varepsilon \le 1 \\ \frac{dz}{dt} &= -\beta y - \gamma z, \end{aligned} \tag{5}$$

Note that for $\varepsilon = 0$, $g(\varepsilon,\ \cdot) \equiv h(\cdot)$, and (5) becomes equivalent to (3). For $\varepsilon = 1$, two cubic segments meet in the origin with C^1-continuity.

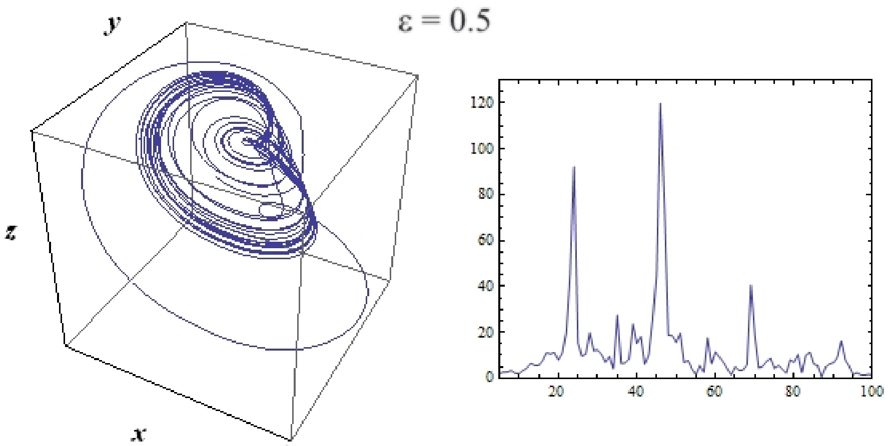

Fig. 3. Attractor of dynamical system (6) with $\varepsilon = 0.5$, and the corresponding Fourier spectrum

3 Numerical experiments

Example 1. In this example, the attractor in the state space $(x,\ \dot{x},\ \ddot{x})$ will be constructed as an approximate solution of the system

$$\begin{aligned} \frac{dx}{dt} &= -6.69191\,(y-x-g(\varepsilon,\ x))\,, \\ \frac{dy}{dt} &= x-y+z, \\ \frac{dz}{dt} &= 1.52061y, \end{aligned} \tag{6}$$

where $g(\varepsilon,\ x)$ is given by (4). This system is of type (3), with $\gamma = 0$. The slopes are set to the prescribed values $m_0 = -8/7$ and $m_1 = -5/7$. The initial values are chosen such that $\{x_0,\ y_0,\ z_0\} = \{-1.4, -0.3, 1.1\}$; The chaos is evident over all the range of $\varepsilon \in [0,\ 1]$, Figs 2 - 4 (left). There are no major differences in the attractors themselves, only the transient trajectories differ noticeably. As expected, the spectra, obtained by Fourier discrete transform reveal more uncommon details (Figs 2 - 4, right). The corresponding Poincaré maps exhibit typical dense set property, characteristic for a chaotic process.

Example 2. Now, the slope values are kept the same $m_0 = -8/7$ and $m_1 = -5/7$, while the initial values are $\{x_0,\ y_0,\ z_0\} = \{0.1,\ 0.1,\ 0.6\}$; The system is of generalized type (5), with $\alpha = 15.6$, $\beta = 27$, which means

$$\begin{aligned} \frac{dx}{dt} &= 15.6\,(y-x-g(\varepsilon,\ x))\,, \\ \frac{dy}{dt} &= x-y+z, \\ \frac{dz}{dt} &= -27y+0.01\,z. \end{aligned} \tag{7}$$

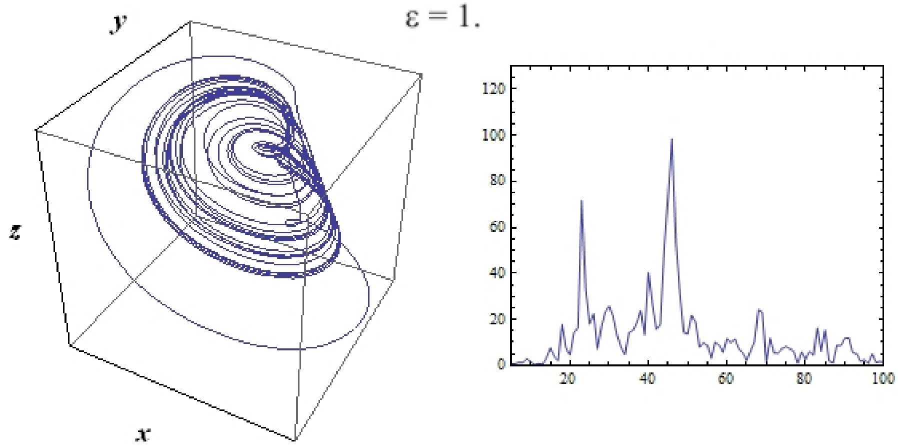

Fig. 4. Attractor of dynamical system (6) with $\varepsilon = 1$, and the corresponding Fourier spectrum

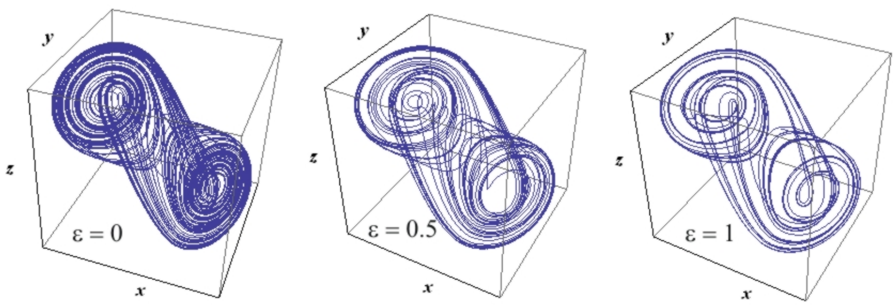

Fig. 5. "Double scroll" attractor for system (7) for various ε;

The attractors are displayed in Figure 5, for three different values of ε. The familiar shape of the “double scroll” alters considerably, indicating that the system may not be sensitive for some choice of system parameters, but for some other it exhibits much more sensibility.

The corresponding Poincaré maps in this case also conform chaotic behavior of the system.

4 Further investigations

The interest in Chua’s dynamical system and its circuit permanently grows. Experts keep finding new applications in different domains. Chua’s circuit is interesting also for the mathematical theory of strange attractors and chaos alone. It exhibits a number of distinct routes to chaos (through e period-doubling scenario, through the breakdown of an invariant torus, etc). The “double scroll” attractor has multi-structural geometry and is more complicated object than any similar attractor of 3D flow, known from literature.

The direction of research anticipated in this note should be deepening by using generalized function $g(\varepsilon,\ x)$. It can be done in many ways, for ex. by introducing non-symmetric ε-neighborhoods of C^0 points, or by introducing higher polynomial degrees or rational functions. Also, the impact of smoothing should be observed on other elements of chaotic dynamics like structurally unstable Poincare homoclinic orbits.

References

[1] ALGABA, A., FREIRE, E., GAMERO, E., RODRIGUEZ-LUIS, A.J., *On the Takens-Bogdanov bifurcation in the Chua's equation*, IEICE Transactions on Fundamentals of Electronics Communications & Computer Sciences, vol. **E82-A**, no.9, (1999), 1722-1728.

[2] BILOTTA, E., PANTANO, P., *Gallery of Chua Attractors*, World Scientific Series on Nonlinear Science, Series A – Vol. **61**, Singapore, 2008.

[3] CHUA, L. O., KOMURO, M., AND MATSUMOTO T., *The double scroll family*, IEEE Trans. Circuit and Systems, **31**, (1986), 1072-1118.

[4] GEGOVSKA-ZAJKOVA, S., KOCIĆ, LJ. M., *Rational Approximated Models Of Chaotic Oscillators*, IV Congress of Mathematicians of Macedonia, October 19-22, 2008, Struga, R. Of Macedonia.

[5] GEGOVSKA-ZAJKOVA, S., KOCIĆ, LJ. M., E., BABACE, *Chaos in Jerk Dynamics*, ETAI Sept. 26-29, 2009, Ohrid.

[6] HIRSCH, M. W., SMALE, S., DEVANEY, R. L., *Differential Equations, Dynamical Systems, And An Introduction To Chaos*, Elsevier Academic Press, 2004.

[7] HUANG, A., PIVKA, L., WU, C-W., FRANZ, M.**,** *Chua's equation with cubic nonlinearity***,** International Journal of Bifurcation & Chaos in Applied Sciences & Engineering, vol. **6**, no.12A, (1996), 2175-2222.

[8] KOCIĆ, LJ. M., GEGOVSKA-ZAJKOVA, S., *A Class Of Chaotic Circuits*, Sedma Nacionalna Konferencija so Meunarodno Uestvo ETAI '2005 (Seventh National Conference with International Participation ETAI'2005) Ohrid, Republic of Macedonia, 21 - 24. IX 2005, Ed. M. J. Stankovski, pp. A.134 – A.138.

[9] KOCIĆ, LJ. M., STEFANOVSKA, L. AND GEGOVSKA-ZAJKOVA, S., *Approximations of Chaotic Circuits*, Proceedings of the 11th Symposium of Mathematics and its Applications, November, 2-5, 2006, (Ed. N. Boja), Timisoara Research Centre of the Romanian Academy and "Politehnica" University of Timisoara, 159-166.

[10] KOCIĆ, LJ. M., STEFANOVSKA, L. AND GEGOVSKA-ZAJKOVA, S., *Approximated Models of Chaotic Oscillators*, Physica Macedonica **56** (2006), 29-36.

[11] KOCIĆ, LJ. M., GEGOVSKA-ZAJKOVA, S., STEFANOVSKA, L., *Numerical Lyapunov Stability*, A4-6, CD- Proceedings of the Eight National Conference with International Participation-ETAI 2007, Ohrid, R. Macedonia, 19-21.IX.2007.

[12] KOCIĆ, LJ. M., GEGOVSKA-ZAJKOVA, S., *On a Jerk Dynamical System*, Facta Univ. Ser. Automatic Control and Robotics (to appear).

[13] KOCIĆ, LJ. M., STEFANOVSKA, L. AND GEGOVSKA-ZAJKOVA, S., *Numerical Examination of Minimal Chaos*, In: Finite Difference Methods: Theory and Applications (Proc. Fourth Int. Conf. FDM:T&A'06, Aug. 26-29, 2006, Lozanetz, Bulgaria, I. Farago, P. Vabishchevich, L. Vulkov, Eds), pp. 234-238, Rousse Univ. "Angel Kanchev", Rousse, 2007.

[14] MATSUMOTO, T., CHUA, L. O., AND KOMURO, M., *The Double Scroll*, IEEE Trans. Circuit and Systems, **32**, (1985), 797-818.

[15] MATSUMOTO, T., CHUA, L. O., AND AYAKI, K., *Reality of Chaos in the Double Scroll Circuit: A Comuter-Assisted Proof*, IEEE Trans. Circuit and Systems, **35**, (1988), 909-925.

[16] PIVKA, L, WU, C-W., HUANG, A., *Chua's oscillator: a compendium of chaotic phenomena*, Journal of the Franklin Institute, vol.**331B**, No.6, (1994), 705-741.

[17] SHIL'NIKOV, L. P., Chua's Circuit: Rigorous Results and Future Problems, IEEE Trans. Circuit and Systems-I: Fundamental Theories and Applications, **40**, No 10, 1993, 784-786.

[18] TANG, K.S., MAN, K.F., *An alternative Chua's circuit implementation*, IEEE International Symposium on Industrial Electronics. Proceedings. ISIE'98 (Cat. No. 98TH8357). IEEE. Part vol.2, (1998), 441-444.

[19] ZHONG, G-Q., *Implementation of Chua's Circuit with a Cubic Nonlinearity*, IEEE Trans. Circuit and Systems-I: Fundamental Theories and Applications, **41**, No 12, 1994, 934-941.