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Position Error Estimation of a Laser Illuminated Object

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Abstract: Position error a laser illuminated object (target) is analyzed. The relations for the displacement signal and probability density function of displacement signal by laser illuminated object are given. The error of displacement signal and the position error are derived. The error of displacement signal of laser illuminated object versus to signal-to-noise ratio (SNR) and the mean value displacement signal. The error of displacement signal is inverse proportional to the root square signal-to-noise ratio, and increases with the mean value displacement signal. The minimum error of displacement signal is root square two times smaller than maximum value, for constant signal-to-noise ratio. The position error is proportional to the error of displacement signal.The minimum position error is obtained when the center of the spot is in the center of quadrant photodiode, and increases with $|x_0|/r$, for constant SNR. The position error rapidly increases at the limits of the measurement range $x_0/r = \pm 1$.

Keywords: quadrant photodiode, displacement signal, probability density function.

1 Introduction

There are a number applications in which accurate positioning are needed, as industry and army [1]. That applications main characteristic is accuracy positioning of an object. Lasers systems positioning have special treatment, because those resolution is bigger than radar and other. A number applications lasers positioning include tracking illuminated target and measurement its angular position [2], [3], estimation vibration effect of Satellite [4], and measurement multidimensional displacement in space [5]. These and other applications can be used position sensitive detectors with quadrant photodiode (QPD) [6] or lateral effect photodiode (LEP) [7] to measure lateral displacement in two perpendicular plane. High-resolution multidimensional displacement monitoring system had development with four quadrant photodiode [5] that can be used monitor six degrees freedom. They have presented results that lateral resolution better than 50nm and angular displacement better than 0.25 micro-radians [5].

Theoretical displacement signal analysis had disused in [1], [2], [3], [4]. In papers is presented that main parameter is signal-to-noise ratio, which limited accuracy of positioning in laser systems with quadrant photodiode. Also, has found that the position error is

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changed with mean value of displacement signal [8]. The expression for probability density function of displacement signal for Gaussian noise distribution is presented in [9].

In this paper a new approach to estimate the position error of laser illuminated object by quadrant photodiode is given. This approach is statistically, where the position error is calculated on base the error of displacement signal, which obtained from probability density function of displacement signal.

2 Displacement signal

The positioning laser illuminated object based on focusing of reflected from object laser beam by receiving optics. Displacement angle between the receiver optical axes and receiving irradiance is shown on Figure 1. The purpose of the optical receiver is to collect the reflected optical energy from the illuminated object.

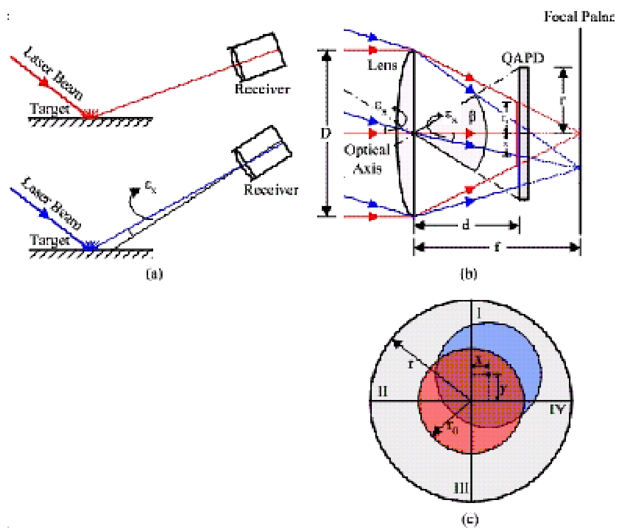

Fig. 1. Laser beam and receiver of reflected laser beam

The plan convex lens, which is placed at the input of the optical receiver, collects the incoming optical energy on the quadrant photodiode (QPD). Figure 2 shows base optical geometry two main components a thin lens, and a quadrant photodiode. The incident reflected laser energy incomes on the thin lens and its focuses on the quadrant photodiode surface. To minimize the error the plan convex a spherical lens is mostly used, because of its minimal spherical aberration coefficient. The QPD is located behind the lens at the distance d from it. Figure 2 gives the example of QPD located in front of focal plane, e.g. $d < f$, where f is the focal length of the lens. As the QPD isn't placed at the focal plane, a light spot with an approximately uniform distribution of irradiance is formed on its surface. In Figure 2 incident radiation incomes with angle δ respect to normal on the surface of lens and produces spot with center (C) on the quadrant photodiode. Cartesian coordinate system sets in center of photodiode, and the center of spot C has coordinates x_0,y_0. Figure 2 shows that point C is function of incident angle δ, for constant distance (d) between thin lens and photodiode. From geometry in Figure 2 can be obtain tg$\delta_h = x_0/d$ i tg$\delta_v = y_0/d$,where δ_h and δ_v are the angles in horizontal and vertical planes, respectively.

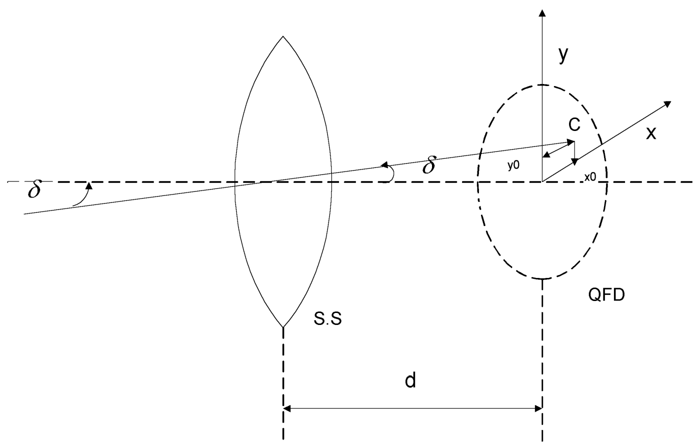

Fig. 2. Geometry between thin lens and quadrant photodiode

A spot position on quadrant photodiode diameter$2a$ shows in Figure 3. The diameter of spot $2r$ is function of distance d and focal length $f$of thin lens.

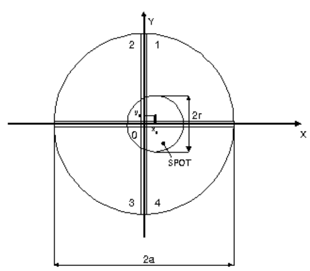

Fig. 3. A spot position on quadrant photodiode surface

The displacement signal in normalized form, for horizontal and vertical plane, from Figure 3, are:

$$\begin{aligned} \varepsilon_x &= \frac{(i_1+i_4)-(i_2+i_3)}{(i_1+i_4+i_2+i_3)} = \frac{i_x}{i_\Sigma} \\ \varepsilon_y &= \frac{(i_1+i_2)-(i_3+i_4)}{(i_1+i_2+i_3+i_4)} = \frac{i_y}{i_\Sigma} \end{aligned} \quad (1)$$

where i_i is the current i-th quadrants (i =1,2,3,4).

The current i-th quadrants is $i_i = \Re p P_0$,where P_0 is the total receiving flux, q is part of one ($1 \geq p \geq 0$), a $\Re$ is the responsivity of photodiode. The total current is the sum of four currents ($i_\Sigma = \Re P_0$).

There are two models for calculation displacement signal as function displacement the center of the spot respect to the center of the photodiode. The first models that assume constant distribution of the irradiance on the sensitive surface photodiode and limited sizes of the spot with circle sharps. Other models assume Gaussian or sinc distribution of the irradiance on the sensitive photodiode surface.

The displacement signal, for constant irradiance distribution on the circle sharp of the spot, from Figure 3, are:

$$\begin{aligned} \varepsilon_x &= \frac{2}{\pi}\left(\frac{x_0}{r}\sqrt{1-\frac{x_0^2}{r^2}}+\arcsin(\frac{x_0}{r})\right), \quad |x_0| \le r \\ \varepsilon_y &= \frac{2}{\pi}\left(\frac{x_0}{r}\sqrt{1-\frac{y_0^2}{r^2}}+\arcsin(\frac{y_0}{r})\right), \quad |y_0| \le r \end{aligned} \tag{2}$$

where are $\frac{x_0}{r}=\frac{dtg\delta_h}{r}, \frac{y_0}{r}=\frac{dtg\delta_v}{r}$.

Center of spot is approximately proportional to incident angle, because $tg\delta \approx \delta$ for small value of δ *(*$\delta \quad \le$ *10*^{0}). The ratio d/r is the constant of the position sensor ($d/r = K_D$).

The displacement signal, for Gaussian distribution of the irradiance with center (x_0,y_0) and square sharp of photodiode size2a, are:

$$\begin{aligned} \varepsilon_x &= \frac{2erf\left(\frac{x_0}{\sqrt{2}\sigma}\right)+erf\left(\frac{a-x_0}{\sqrt{2}\sigma}\right)-erf\left(\frac{a+x_0}{\sqrt{2}\sigma}\right)}{erf\left(\frac{a-x_0}{\sqrt{2}\sigma}\right)+erf\left(\frac{a+x_0}{\sqrt{2}\sigma}\right)} \\ \varepsilon_y &= \frac{2erf\left(\frac{y_0}{\sqrt{2}\sigma}\right)+erf\left(\frac{a-y_0}{\sqrt{2}\sigma}\right)-erf\left(\frac{a+y_0}{\sqrt{2}\sigma}\right)}{erf\left(\frac{a-y_0}{\sqrt{2}\sigma}\right)+erf\left(\frac{a+y_0}{\sqrt{2}\sigma}\right)} \end{aligned} \tag{3}$$

where are: σ- is the standard deviation of two-dimensional Gaussian distribution irradiance on the sensitive photodiode surface, and *erf(x)* is the error function, where $erf(x) = \frac{2}{\pi}\int_0^x \exp(-u^2)du$.

The standard deviation σ in (3) and the radius of spot r in (2) is in relationship. The part of flux in the circle radius r respect to total incident flux is

$$\frac{P}{P_T} = \frac{1}{2\pi\sigma^2}\int_0^{2\pi} d\varphi \int_0^r \exp(-\frac{\rho^2}{2\sigma^2})\rho d\rho = 1-\exp(-\frac{r^2}{2\sigma^2}) \tag{4}$$

For example, from (4), ratio P/P_T=90% of total flux lies in circle of radius r=2.1456σ.

The diagrams of the displacement signal from (2), (3) are given on Figure 4, for r=2.1456σ, (r= 2.25 mm, σ =1.048 mm). The best agreement displacement signal for Gaussian and circular spot on the sensitive surface QPD is found around x=0 and x/r=±1, as shown on Figure 4.

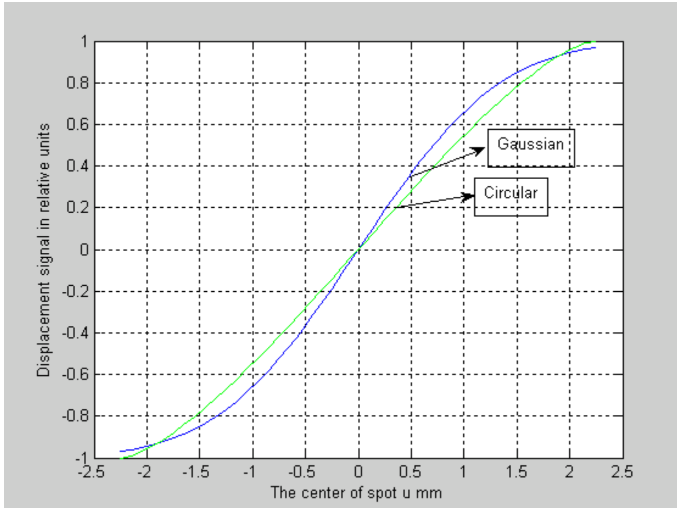

Fig. 4. Displacement signal ε_x for constant irradiance and Gaussian distribution irradiance on the surface photodiode, for r=2.1456σ, (r= 2.25 mm, σ =1.048 mm)

3 Error of displacement signal and position error

Expressions for the displacement signal (3) and (4) are derived with assumptions for the spot sharp geometry and the irradiance distribution. Displacement signal in real is sum of signal and noise. The noise add to signal influence to changing center of spot, which play around $x_{0,}y_0$. The position error can be understood as the changing of the center of spot, which can not be detected. Those changing of the center of spot Δx_0 and Δy_0 can be estimate from expressions for the displacement signal. The error of the displacement signal from (1) for one axis is

$$\Delta\varepsilon_x = \frac{\Delta i_x}{i_\Sigma} \tag{5}$$

where i_Σ and r are constant.

We assumed that the current difference is equal the standard deviation of the total noise current (Δi_x=σ_t). Now from (5) the error of displacement signal $\Delta\varepsilon_x$ for x-axis is

$$\Delta\varepsilon_x = \frac{1}{\sqrt{SNR}} \tag{6}$$

where SNR is the signal-to-noise ratio in the channel sum.

Equations (6) shows that the error displacement signal of laser illuminated object is inverse proportional to root square of signal-to-noise ratio, in channel sum.

The error of displacement signal obtained from the derivative of the displacement signal (2), for x-axis is

$$\Delta\varepsilon_x = \frac{4}{\pi}\frac{\Delta x_0}{r}\sqrt{1-(x_0/r)^2} \tag{7}$$

where Δx is the position error.

Equation (7) shows that the position error (Δx_0) is proportional to the error of displacement signal ($\Delta\varepsilon_x$).

The displacement signal ε_x and ε_y in (1) are combination current of incident irradiance and noise of photodiode. The displacement signal (1), for one axes can be written as:

$$\varepsilon = \frac{S_D + N_1}{S_S + N_2} = \frac{u - v}{u + v} \tag{8}$$

where

S_D –is the pair wise difference signal for a given axis,

S_S – is the sum signal all (four) quadrants,

N_1 – is the root mean square (rms) noise associated difference signal,

N_2 – is the rms noise of the sum signal.

u, v- are the signal plus noise of sum two quadrants, respectively. We assume that each quadrant generate noise with Gaussian distribution with zero mean value and variance σ_n^2. Then u and v from (8) represents signal from pair quadrants of photodiode

$$\begin{aligned} u &= \bar{u} + N_u \\ v &= \bar{v} + N_v \end{aligned} \tag{9}$$

where $\bar{u}$and $\bar{v}$ are mean value, and $N_u$and $N_v$are fluctuation of$u$and v, respectively.

Probability density function (*pdf*) of displacement signal is obtained, on base known theory for jointly normal random variables u and v, probability density functions $f(u,v)$and transformation its in $f(\varepsilon)$ [10], in the form

$$f(\varepsilon) = \frac{\sqrt{1-\rho^2}}{\pi(1+\varepsilon^2 - 2\rho\varepsilon)} \cdot \exp\left(-\frac{\bar{u}^2 + \bar{v}^2 - \rho(\bar{u}^2 - \bar{v}^2)}{2\sigma_p^2(1-\rho^2)}\right) \cdot \left[1 + \sqrt{\frac{\pi}{2}} \cdot B\left(erf(\frac{B}{\sqrt{2}})\right)\exp\frac{B^2}{2}\right] \tag{10}$$

where are: ρ is the correlation between $N_1 = N_u - N_v$ and $N_2 = N_u + N_v$, σ_p^2=$2\sigma_n^2$, and Bdefines as

$$B = \frac{\bar{u}\,(1+\varepsilon) + \bar{v}\,(1-\varepsilon) - \rho\left(\bar{u}\,(1+\varepsilon) + \bar{v}\,(\varepsilon - 1)\right)}{\sqrt{2}\sigma_p\sqrt{(1-\rho^2)(1+\varepsilon^2 - 2\rho\varepsilon)}}$$

Analysis of *pdf* has given in [11] where it was shown that the correlation coefficient change both the maximum and width of *pdf*. Maximum width and minimum amplitude of *pdf* was obtained for correlation coefficient equal zero. That means the worst case for the error of displacement signal is when uncorrelated noise between pair quadrants (ρ=0).

Useful way to analyze *pdf* of displacement signal $f(\varepsilon)$ that the mean value pair (u, v) substitute with mean value of the displacement signal ($\bar{\varepsilon}$).The mean value of displacement signal from (8) and (9) is:

$$\bar{\varepsilon} = \frac{\bar{u} - \bar{v}}{\bar{u} + \bar{v}} \tag{11}$$

Signal-to-noise ratio in the channel sum is

$$SNR = \frac{(\bar{u}+\bar{v})^2}{4\sigma_n^2} \tag{12}$$

After substitute (11) and (12) in (10) and arrangement equation for B for ρ=0 becomes

$$B = \frac{\bar{u}(1+\varepsilon)+\bar{v}(1-\varepsilon)}{\sqrt{2}\sigma_p\sqrt{1+\varepsilon^2}} = \frac{1+\varepsilon\bar{\varepsilon}}{\sqrt{1+\varepsilon^2}}\sqrt{SNR} \tag{13}$$

Used to (11) and (12) can be written

$$\frac{\bar{u}^2+\bar{v}^2}{4\sigma_n^2} = \frac{1}{2}SNR(1+\bar{\varepsilon}^2) \tag{14}$$

Probability density function (10) for uncorrelated noise (ρ=0) as function the mean value of displacement signal is given on Fige 5.

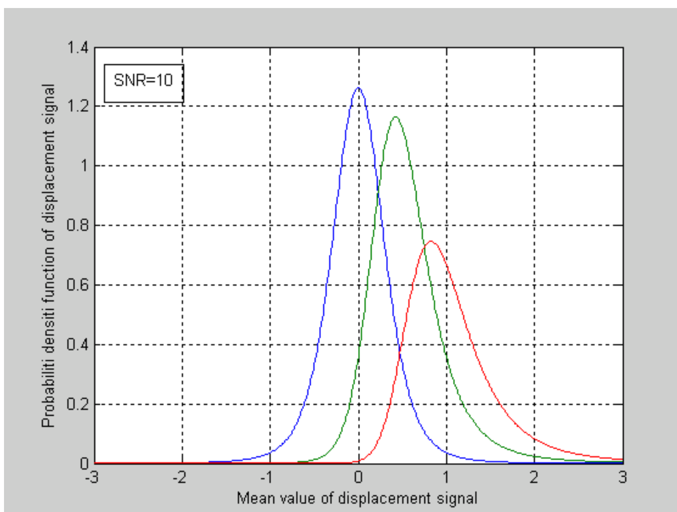

Fig. 5. Probability density function of displacement signal as function for$\bar{\varepsilon}$=0.0, 0.5 and 1, and SNR=10

From Figure 5 can be seen changes both the maximum and width of *pdf*. Maximum width and minimum amplitude of *pdf* was obtained for $\bar{\varepsilon}$=1.

Probability density function of displacement signal from (10), for ρ=0, after applied (13) and (14) and approximation for the complementary error function erfc(x)≈exp(-x^2)/(x$\pi^{1/2}$), becomes

$$f(\varepsilon) \approx \sqrt{\frac{SNR}{2\pi}}\frac{1+\varepsilon\bar{\varepsilon}}{(1+\varepsilon^2)^{3/2}}\exp\left(-\frac{SNR(\varepsilon-\bar{\varepsilon})^2}{2(1+\varepsilon^2)}\right) \tag{15}$$

Probability density function (15) is good approximation *pfd* (10) for uncorrelated noise. The error of this approximation is smaller than 0.8% for the worst case: $\bar{\varepsilon}$=0, and small SNR=5,

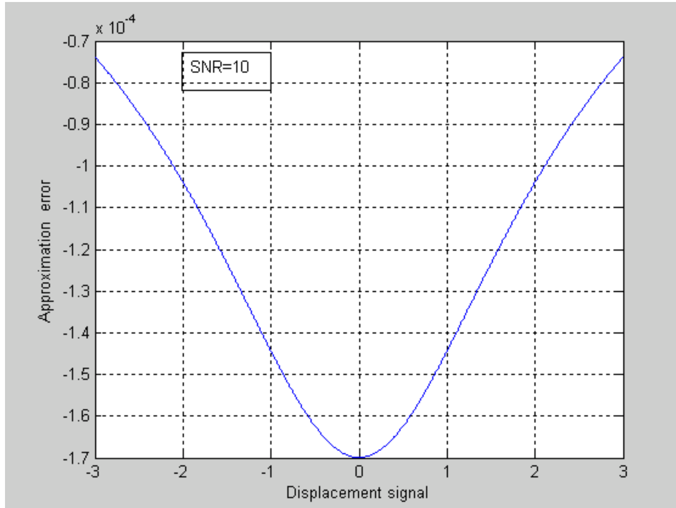

Fig. 6. Approximation error for SNR=10 and $\bar{\varepsilon}$ =0.

and descries very fast with increases SNR. In Figure 6 is shown approximation error versus displacement signal ε, for mean value $\bar{\varepsilon}$ and signal-to-noise SNR as parameters.

From approximation form *pdf* of displacement signal (15) can be derived the error of displacement signal. The error of displacement signal presented small range $\Delta\varepsilon$ around the mean value $\bar{\varepsilon}$ in which probability is required. This range $\Delta\varepsilon$ usually calculates for probability 50% (CEP-the Circular Error Probability). Probability positioning is given by

$$P_p = \int_{\vec{\varepsilon}-\Delta\varepsilon}^{\bar{\varepsilon}+\Delta\varepsilon} f(\varepsilon)d\varepsilon \tag{16}$$

where $\Delta\varepsilon$ is the error of displacement signal.

Integral (16) can will solved in closed form for probability density function is given (15). After substitution $\frac{(\varepsilon-\bar{\varepsilon})^2}{1+\varepsilon^2} = x$ integral from (16) in the unlimited form becomes

$$\int f(\varepsilon)d\varepsilon = \int \sqrt{\frac{SNR}{2\pi}} \frac{1}{2\sqrt{x}} \exp\left(-\frac{SNR}{2}x\right) dx = \frac{1}{2} erf\left(\sqrt{\frac{SNR}{2}} \frac{\varepsilon-\bar{\varepsilon}}{\sqrt{1+\varepsilon^2}}\right) \tag{17}$$

From (16), and (17) probability positioning that the displacement signal ε lies in range $\Delta\varepsilon$ around$\bar{\varepsilon}$ becomes

$$P_p = \frac{1}{2}\left[erf\left(\frac{\sqrt{SNR}}{\sqrt{2}} \frac{\Delta\varepsilon}{\sqrt{1+(\bar{\varepsilon}+\Delta\varepsilon)^2}}\right) + erf\left(\frac{\sqrt{SNR}}{\sqrt{2}} \frac{\Delta\varepsilon}{\sqrt{1+(\bar{\varepsilon}-\Delta\varepsilon)^2}}\right)\right] \tag{18}$$

It can be shown, that for SNR¿10, the arguments of the error functions allow further approximation of P_p from (3. 14) leading to

$$P_p \approx erf\left(\sqrt{\frac{SNR}{2}} \frac{\Delta\varepsilon}{\sqrt{1+\bar{\varepsilon}^2+\Delta\varepsilon^2}}\right) \tag{19}$$

The relative error of this approximation, is the worst case $\bar{\varepsilon}$=1, is less than 2,5%. By solving (19) for $\Delta\varepsilon$ we obtain the error of displacement signal is

$$\Delta\varepsilon = \frac{\sqrt{1+\bar{\varepsilon}^2}}{\sqrt{(SNR/2C^2)-1}} \tag{20}$$

where C =erfinv(P_p).

As can be seen from (20), the maximum value of $\Delta\varepsilon$, for any given signal-to-noise ratio is $\sqrt{2}$ times larger for$\bar{\varepsilon}$=1 in comparison to the minimum value for $\bar{\varepsilon}$=0. The minimum value of $\Delta\varepsilon$ is inversely proportional to$\sqrt{SNR}$, for $SNR >> 2C^2$, as obtained in (2). The error of displacement signal (20), for $SNR >> 2C^2$becomes

$$\Delta\varepsilon \approx \frac{\sqrt{2}C}{\sqrt{SNR}}\sqrt{1+\bar{\varepsilon}^2} \tag{21}$$

where an approximation error is smaller than 10^{-3}.

In Figure 7 is shown the error of displacement signal as function of the mean value of the displacement signal, and SNR as parameter.

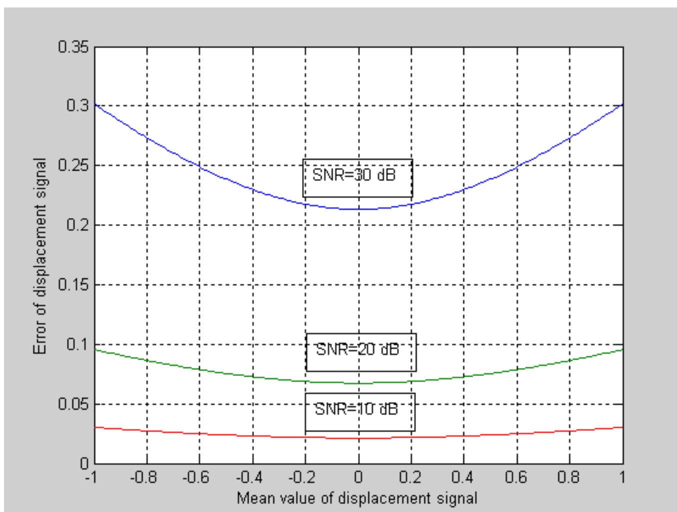

Fig. 7. The error of displacement signal as function mean value, for Pp=50% and SNR=10 dB, 20 dB, and 30 dB.

The error of displacement signal changes with SNR and the mean value of the displacement signal as shows in Figure 7. The error of displacement signal always is minimum for the mean value displacement signal equal zero and maximum for the mean value equal plus or minus one. An example, the error of displacement signal from (21) and probability positioning 50% (CEP), lies in range between minimum value $\Delta\varepsilon$=0.675/$\sqrt{\text{SNR}}$, for $\bar{\varepsilon}$=0, and maximum value $\Delta\varepsilon$=0,954/$\sqrt{\text{SNR}}$, $\bar{\varepsilon} = \pm 1$.

Now we have two results for the error of displacement signal. The first result (6) is obtained directly from the displacement signal, and second result (21) is derived from probability density function of the displacement signal. Both results show that the error of displacement signal a laser illuminated target is inverse proportional to root square signal-to-noise ratio. The error of displacement signal given in (6) remained constant for given SNR, but the error displacement signal (21) changes depend on the mean value of displacement signal.

The error of displacement signal $\Delta\varepsilon$ and the position error Δx are proportional as shows in (3). The normalized position error in the x direction, from (3) is

$$\frac{\Delta x_0}{r} = \frac{\pi}{4} \frac{\Delta\varepsilon}{\sqrt{1-\frac{x_0^2}{r^2}}} \tag{22}$$

where $\Delta\varepsilon$ is the error of displacement signal (21).

Normalized position error calculated from (22) and (21) as function normalized position x_0/r, is shown in Figure 8.

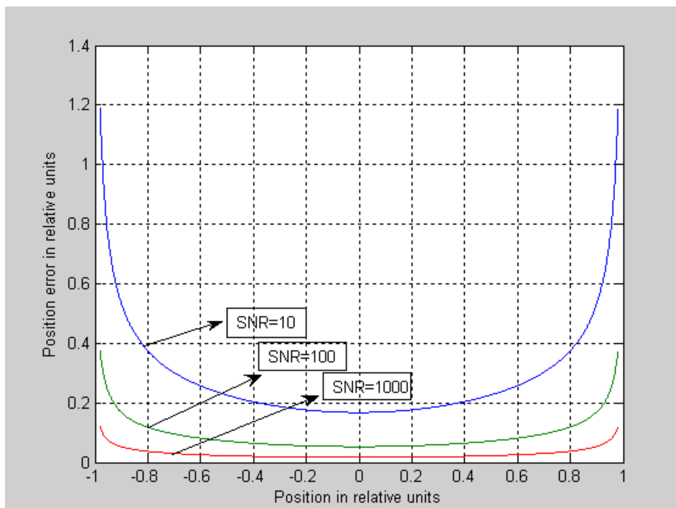

Fig. 8. Normalized position error as function normalized position for P_p=0.5, and SNR=10, 100 and100

From Figure 8, may be seen that the normalized position error depends on SNR and normalized position x_0/r. The minimum position error is obtained when the center of the spot is in the center of quadrant photodiode, and increases with $|x_0|$/r, for constant SNR. The position error rapidly increases at the limits of the measurement range x_0/r=$\pm$1, as shows Figure 8.

4 Conclusion

The error displacement signal inverses proportional to the root square of signal-to-noise ratio. For the mean value of displacement signal equal zero and probability 50% the error of displacement signal is 0.675/$\sqrt{\text{SNR}}$. The error of displacement small increase with grows the mean value of displacement signal. The maximum value of the error of displacement signal is $0.954/\sqrt{SNR}$, on the ends of range positioning. Maximum value of the error of displacement signal is root square times bigger than minimum value, for constant both probability and signal-to-noise ratio.

The position error depends on SNR and normalized position x_0/r. The minimum position error is obtained when the center of the spot is in the center of quadrant photodiode, and increases with $|x_0|$, for constant SNR. The position error rapidly increases at the limits of the measurement range $x_0/r=\pm 1$.

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