

Vibrations of a System: Viscoelastic Rod of Fractional Type and a Body Attached to the Rod

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Abstract: We consider motion of a system consisting of a viscoelastic rod of finite length, described by the fractional Zener model and a body attached to its free end (the other end of a rod is fixed). The motion of a rod is described by the system of partial differential equations of integer and fractional order. The motion of a body is translatory along the axis of the rod and it represents the boundary condition for the equations describing the motion of a rod. The initial-boundary value system is solved by the use of the Laplace transform and is given in the convolution form. Solutions are tested numerically.

Keywords: fractional derivative, viscoelastic material of fractional Zener type, vibrations of a rod, vibrations of a body

1 Introduction

We treat the motion of a rod with one end fixed, while attached at the other end there is a body of mass m . The length of a rod is L and its axis, at initial time moment as well as during the motion, coincides with the $\bar{x}$ axis. Let x denote a position of a material point of a rod at the initial time instant $t_0 = 0$. The position of this point at time instant t is $x + u(x, t)$. Equations describing the motion are ($x \in [0, L], t > 0$)

$$\frac{\partial}{\partial x} \sigma(x, t) = \rho \frac{\partial^2}{\partial t^2} u(x, t), \quad \varepsilon(x, t) = \frac{\partial}{\partial x} u(x, t), \quad (1)$$

$$(1 + a_0 D_t^\alpha) \sigma(x, t) = E (1 + b_0 D_t^\alpha) \varepsilon(x, t), \quad (2)$$

$$u(x, 0) = 0, \quad \frac{\partial}{\partial t} u(x, 0) = 0, \quad \sigma(x, 0) = 0, \quad \varepsilon(x, 0) = 0, \quad (3)$$

$$u(0, t) = 0, \quad -A \sigma(L, t) + F(t) = m \frac{\partial^2}{\partial t^2} u(L, t). \quad (4)$$

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In equation of motion of the rod and in the strain measure (1) we used σ , u and ε to denote stress, displacement and strain, respectively, of a material point as unknown functions depending on initial position x and time t . In (1) ρ denotes the density of the material. The constitutive equation (2) is the fractional Zener model of a solid-like viscoelastic material, E is a generalized Young modulus (positive constant having dimension of stress), a and b are generalized relaxation times satisfying $a \leq b$, restriction obtained from the Second Law of Thermodynamics. The operator of the left Riemann-Liouville fractional derivative of order $\alpha \in (0, 1)$ ${}_0D_t^\alpha$ is defined as

$${}_0D_t^\alpha y(t) := \frac{d}{dt} \left(\frac{t^{-\alpha}}{\Gamma(1-\alpha)} * y(t) \right), \quad t > 0,$$

where Γ is the Euler gamma function, $*$ is a convolution (if $f, g \in L_{loc}^1(\mathbb{R})$, $\text{supp } f, g \subset [0, \infty)$, then $(f * g)(t) := \int_0^t f(\tau) g(t - \tau) d\tau$, $t \in \mathbb{R}$). We refer to [8, 10, 16] for a detailed account on fractional calculus. Initial conditions (3) specify that the system rod-body is in the rest in the initial time instant. Boundary condition (4)₁ imply that one end of a rod is fixed. The other boundary condition (4)₂ describes the translatory motion, along the $\bar{x}$ axis, of a body attached to the free end of a rod. In (4) A stands for the cross-section area of the rod and F for the known external force acting on the body.

Regarding the constitutive equation (2) we note that it describes the solid-like viscoelastic material of fractional-order type. The difference between solid and fluid-like materials is observed in the creep test, i.e., when material is subject to a sudden, but later constant force on its free end. Namely, solid-like materials creep to a finite value of displacement, in contrast to the fluid-like materials that creep to an infinite value of displacement. In [5] we have analyzed the creep test of a material described by the constitutive equation

$$\int_0^1 \phi_\sigma(\gamma) {}_0D_t^\gamma \sigma(x, t) d\gamma = \int_0^1 \phi_\varepsilon(\gamma) {}_0D_t^\gamma \varepsilon(x, t) d\gamma, \quad (5)$$

with constitutive functions $\phi_\sigma(\gamma) := a^\gamma$ and $\phi_\varepsilon(\gamma) := b^\gamma$, $\gamma \in (0, 1)$, $a \leq b$ (see [1, 2, 4]), which is a generalization of a linear fractional-type constitutive equation (2). The conclusion was (according to numerical examples) that the material is solid-like. In [4] we have analyzed the same problem with the constitutive distribution, proposed in [17] and given by $\phi_\sigma(\gamma) := \delta(\gamma) + a\delta(\gamma - \alpha)$ and $\phi_\varepsilon(\gamma) := b_0\delta(\gamma - \beta_0) + b_1\delta(\gamma - \beta_1) + b_2\delta(\gamma - \beta_2)$, where δ is the Dirac delta distribution, $a, b_0, b_1, b_2 > 0$, $0 < \alpha < \beta_0 < \beta_1 < \beta_2 \leq 1$, and the conclusion was (again based upon numerical examples) that the material is fluid-like. We note that problems similar to (1) - (4) were also treated in [13] for the constitutive equation $(1 + \tau_\varepsilon^\alpha {}_0D_t^\alpha) \sigma = E_\infty \tau_\varepsilon^\beta {}_0D_t^\beta \varepsilon$ and in [12] for the constitutive equation $(1 + \tau_\varepsilon^\alpha {}_0D_t^\alpha) \sigma = E_0 \left(1 + \tau_\sigma^\alpha {}_0D_t^\alpha + \tau_\sigma^\beta {}_0D_t^\beta \right) \varepsilon$. For the detailed account of applications of fractional calculus in viscoelasticity see [9, 11, 15].

2 Convolution form of the solution

The system (1) - (4) transforms into $(x \in [0, 1], t > 0)$

$$\frac{\partial}{\partial x} \sigma(x, t) = \kappa^2 \frac{\partial^2}{\partial t^2} u(x, t), \quad \varepsilon(x, t) = \frac{\partial}{\partial x} u(x, t), \quad (6)$$

$$(1 + a_0 D_t^\alpha) \sigma(x, t) = (1 + b_0 D_t^\alpha) \varepsilon(x, t), \quad (7)$$

$$u(x, 0) = 0, \quad \frac{\partial}{\partial t} u(x, 0) = 0, \quad \sigma(x, 0) = 0, \quad \varepsilon(x, 0) = 0, \quad (8)$$

$$u(0, t) = 0, \quad -\sigma(1, t) + F(t) = \frac{\partial^2}{\partial t^2} u(1, t), \quad (9)$$

after introduction of the ratio between the masses of a rod and a body κ , as well as dimensionless quantities

$$\begin{aligned} \kappa &= \sqrt{\frac{\rho AL}{m}}, \quad \bar{x} = \frac{x}{L}, \quad \bar{t} = t \sqrt{\frac{AE}{mL}}, \quad \bar{u} = \frac{u}{L}, \quad \bar{\sigma} = \frac{\sigma}{E}, \\ \bar{a} &= a \left(\sqrt{\frac{AE}{mL}} \right)^\alpha, \quad \bar{b} = \left(\sqrt{\frac{AE}{mL}} \right)^\alpha, \quad \bar{F} = \frac{F}{AE}. \end{aligned}$$

Also in writing (6) - (9) we omitted bar over dimensionless quantities.

In order to solve the system (6), (7) subject to initial (8) and boundary data (9), we use the Laplace transform method. Recall that the Laplace transform of $f \in L_{loc}^1(\mathbb{R})$, $f \equiv 0$ in $(-\infty, 0]$ and $|f(t)| \leq ce^{kt}$, $t > 0$, for some $k > 0$, is defined by

$$\tilde{f}(s) = \mathcal{L}[f(t)](s) := \int_0^\infty f(t) e^{-st} dt, \quad \operatorname{Re} s > k$$

and analytically continued into the appropriate domain D .

Applying formally the Laplace transform to (6) - (9) we obtain $(x \in [0, 1], s \in \mathbb{C} \setminus (-\infty, 0])$

$$\begin{aligned} \frac{\partial}{\partial x} \tilde{\sigma}(x, s) &= \kappa^2 s^2 \tilde{u}(x, s), \quad \tilde{\varepsilon}(x, s) = \frac{\partial}{\partial x} \tilde{u}(x, s), \\ (1 + as^\alpha) \tilde{\sigma}(x, s) &= (1 + bs^\alpha) \tilde{\varepsilon}(x, s), \\ \tilde{u}(0, s) &= 0, \quad \tilde{\sigma}(1, s) + s^2 \tilde{u}(1, s) = \tilde{F}(s). \end{aligned} \quad (10)$$

In order to obtain the displacement u we use (10), so that we have $(x \in [0, 1], s \in \mathbb{C} \setminus (-\infty, 0])$

$$\frac{\partial^2}{\partial x^2} \tilde{u}(x, s) - \kappa^2 s^2 M^2(s) \tilde{u}(x, s) = 0, \quad (11)$$

where we introduce

$$M(s) := \sqrt{\frac{1 + as^\alpha}{1 + bs^\alpha}}, \quad s \in \mathbb{C} \setminus (-\infty, 0]. \quad (12)$$

The solution of (11), subject to (10)₃, is

$$\tilde{u}(x, s) = \tilde{F}(s) \tilde{P}(x, s), \quad x \in [0, 1], \quad s \in \mathbb{C} \setminus (-\infty, 0]. \quad (13)$$

where $(x \in [0, 1], s \in \mathbb{C} \setminus (-\infty, 0])$

$$\tilde{P}(x, s) = \frac{1}{s} \frac{M(s) \sinh(\kappa x s M(s))}{s M(s) \sinh(\kappa s M(s)) + \kappa \cosh(\kappa s M(s))}. \quad (14)$$

3 Explicit form of the solution

In order to calculate P and obtain the displacement u , we have to invert the Laplace transform in (14). The following proposition establishes location and multiplicity of the poles of $\tilde{P}$, given by (14).

Proposition 1 *Let $(s \in \mathbb{C} \setminus (-\infty, 0])$*

$$f(s) := s M(s) \sinh(\kappa s M(s)) + \kappa \cosh(\kappa s M(s)) = 0. \quad (15)$$

- (i) *There are countably many complex conjugated zeros of f , denoted by s_n and $\bar{s}_n$, $n \in \mathbb{N}$, respectively.*
- (ii) *The zeros s_n (and $\bar{s}_n$) of f are located in the open left complex plane for all $n \in \mathbb{N}$.*
- (iii) *The zeros s_n (and $\bar{s}_n$) of f are of multiplicity one, for fixed and sufficiently large $n \in \mathbb{N}$.*

Now we give the theorem on existence and properties of the displacement u .

Theorem 2 *Let F be a locally integrable function on $\mathbb{R}$ equal to zero on $(-\infty, 0)$. The displacement u as a solution to (6) - (9) is given by*

$$u(x, t) = F(t) * P(x, t), \quad x \in [0, 1], \quad t > 0.$$

The function P is determined by $(x \in [0, 1], t > 0)$

$$\begin{aligned} P(x, t) = & \frac{1}{2\pi i} \int_0^\infty \left(\frac{1}{D(qe^{-i\pi})} M(qe^{-i\pi}) \sinh(\kappa x q M(qe^{-i\pi})) \right. \\ & \left. - \frac{1}{D(qe^{i\pi})} M(qe^{i\pi}) \sinh(\kappa x q M(qe^{i\pi})) \right) \frac{e^{-qt}}{q} dq \\ & + \sum_{n=1}^\infty [\text{Res}(\tilde{P}(x, s) e^{st}, s_n) + \text{Res}(\tilde{P}(x, s) e^{st}, \bar{s}_n)], \end{aligned} \quad (16)$$

where M is given by (12) and

$$D(qe^{\pm i\pi}) = q M(qe^{\pm i\pi}) \sinh(\kappa q M(qe^{\pm i\pi})) + \kappa \cosh(\kappa q M(qe^{\pm i\pi})).$$

The residues are given by

$$\text{Res}(\tilde{P}(x, s)e^{st}, s_n) = \left[\frac{M(s) \sinh(\kappa x s M(s))}{s \frac{d}{ds} f(s)} e^{st} \right]_{s=s_n}, \quad (17)$$

where f is given by (15) and s_n , $n \in \mathbb{N}$, are zeros of (15). Function u is continuous on $[0, 1] \times [0, \infty)$.

For the proofs of Proposition 1 and Theorem 2 we refer to [6, 7].

4 Numerical examples

The aim of this section is the qualitative analysis of the behavior of a displacement u obtained as a solution to (6) - (9) in the special case of force F . The existence and the form of u are given in Theorem 2. In order to study creep, we chose the force, acting on the body attached to the rod's end, to be the Heaviside function, i.e. $F = H$. We fix the parameters describing the rod: $a = 0.1$, $b = 0.9$, $\alpha = 0.8$ and give plots of u for different values of the ratio between the masses of rod and particle $\varkappa := \kappa^2$.

Figures 1 and 2 present the long-time behavior of displacement u for different values of $\varkappa$. It is evident that the oscillations of the system rod-body are damped, since the material is viscoelastic. One sees that there is no dependence of the finite value of displacement in creep on the value of the mass ratio $\varkappa$, i.e. for any $\varkappa$ if $F = H$ then $\lim_{t \rightarrow \infty} u(x, t) = x$, $x \in [0, 1]$.

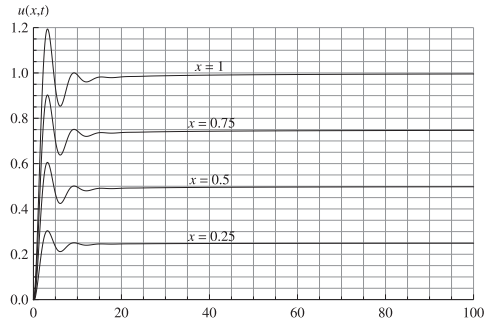


Fig. 1. Displacement $u(x, t)$ in the creep experiment for $\varkappa = 0.5$ as a function of time t at $x \in \{0.25, 0.5, 0.75, 1\}$ for $t \in (0, 100)$.

Figures 3 and 4 present the short-time behavior of u for different values of $\varkappa$. We see that for small times there is an influence of $\varkappa$ on the height of the peaks such that its height increases as $\varkappa$ decreases. However, this is due to the inertia of the body, while for larger times the viscoelastic properties of the rod prevails. There is also an influence of $\varkappa$ on the width of the peaks, since as $\varkappa$ increases the width of the peak does it also. We see that regardless of the value of $\varkappa$ there is a delay in the starting time-instant of oscillations. This is due to the finite wave propagation speed, since the particle ($x = 1$), which is subject to the

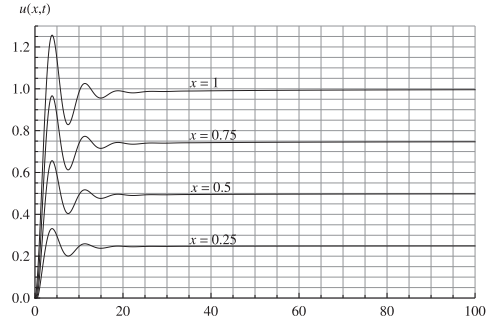


Fig. 2. Displacement $u(x,t)$ in the creep experiment for $\varkappa = 2$ as a function of time t at $x \in \{0.25, 0.5, 0.75, 1\}$ for $t \in (0, 100)$.

action of the force, starts oscillating at $t = 0$, while the starting time-instant of the points of the rod is greater as the point is further from the point where the force acts. The delay in the oscillations starting time-instant increases as $\varkappa$ increases.

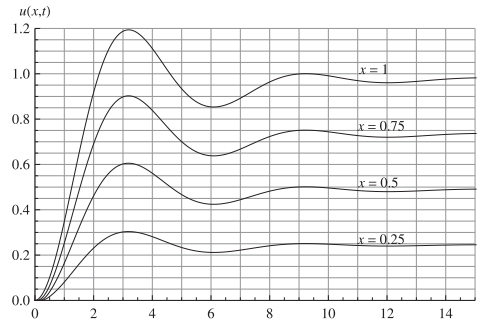


Fig. 3. Displacement $u(x,t)$ in the creep experiment for $\varkappa = 0.5$ as a function of time t at $x \in \{0.25, 0.5, 0.75, 1\}$ for $t \in (0, 15)$.

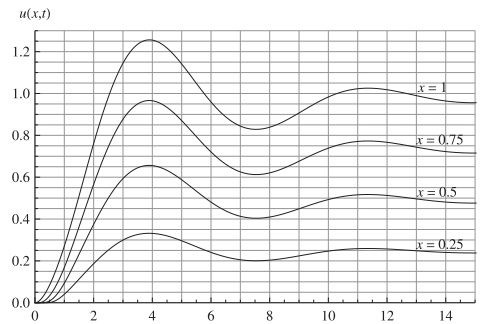


Fig. 4. Displacement $u(x,t)$ in the creep experiment for $\varkappa = 2$ as a function of time t at $x \in \{0.25, 0.5, 0.75, 1\}$ for $t \in (0, 15)$.

Figure 5 presents the plot of time evolution of displacement u for fixed point of the rod $x = 0.5$ if the ratio between masses $\varkappa$ varies. Here, the parameters are: $a = 0.2$, $b = 0.6$,

$\alpha = 0.45$. Figure 5 supports previously mentioned fact that regardless of the $\varkappa$ rod creeps to the same value of displacement.

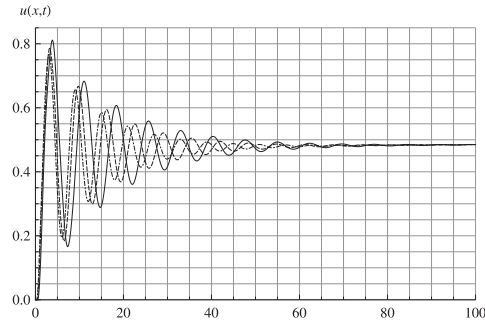


Fig. 5. Displacement $u(x,t)$ in the creep experiment at $x = 0.5$ as a function of time $t \in (0, 100)$ for $\varkappa = 0.5$ - dot-dashed line, $\varkappa = 1$ - dashed line and $\varkappa = 2$ - solid line.

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