SCIENTIFIC PUBLICATIONS OF THE STATE UNIVERSITY OF NOVI PAZAR
SER. A: APPL. MATH. INFORM. AND MECH. vol. 16, 1, 2024, 7–29

A New Optimization Approach for the Tallest Column Design

Dragan T. Spasić, Teodor M. Atanacković

Dedicated to the memory of Professor Ćemal Dolićanin (1945–2023)

Abstract: Motivated by the classical brachistochrone we present a new pattern of finding the shape of a vertical column that attains the maximum height if its material and volume are prescribed. It comprises the optimal control problem with a free end point. Besides, the constitutive equation of the column is such that it can suffer flexure, compression and shear. The critical load of a heavy compressed column for the finite values of shear and extensional rigidity and the novel use of the Pontryagin maximum principle with the corresponding first integral yielding the non-vanishing optimal cross section as a solution of a quadratic equation are main novelties of this work. The classical solution for the tallest column under selfweight is covered as a special case for infinite values of shear and extensional rigidity.

Keywords: tallest column, generalized elastica with shear and axial strain, optimal control problem with a free end point.

1 Introduction

Nonuniformity in formulating optimal control problems is a well known fact. A typical example is the classical brachistochrone with different expressions of time to be minimized and different control variables of either geometrical or physical nature, yielding the same solution. Regarding the tallest column problem in a constant gravity field one can either fix the volume and height of the column and maximize the lowest eigenvalue of the corresponding equations describing the equilibrium configuration, or fix the eigenvalue and height and minimize the volume, for example see [27], [36], [15], [32], [21], or [35], [18], [19] for the first, and [25], [39], [6], [4], for the second approach. In both formulations once the critical load and/or the corresponding volume are determined, one has to resize the column i.e. to calculate the new shape as an effect of the different material distribution along the

Manuscript received December 27, 2023; accepted February 5, 2024

Dragan T. Spasić (ORCID 0000-0003-1848-5379) is with the University of Novi Sad, Department of Applied Mechanics, Novi Sad, Serbia;

Teodor M. Atanacković (ORCID 0000-0002-8714-1388) is with Serbian Academy of Sciences and Arts, Serbia

https://doi.org/10.46793/SPSUNP2401.007S

column axis, but the height of the column is known because it is specified at the beginning. Namely, the height of the column is proportional to the fourth root of the critical load multiplied by the volume. Motivated by the classical brachistochrone problem formulated as a time optimal problem, see [26], in this work we intend to reformulate the tallest column problem. In doing so we are going to fix both: the eigenvalue and the volume. Namely, we shall determine the eigenvalue from the stability condition of the column with constant cross-section made of the same material that is used for the column that will be optimized. Then for this eigenvalue (load) in equilibrium equations we impose the isoperimetric constraint to fix the volume and in turn weight to unity, and then maximize the height of the column as a solution of the optimal control problem with a free end point.

We shall consider the column that is clamped at the base (lower point) and free at the top. This type of boundary conditions, for the Bernoulli-Euler column, leads to the singularity in the differential equations describing equilibrium of the column, see [27], since the cross-section of the tallest column vanishes at the top. In the analysis that follows this will be avoided by imposing a small concentrated force at the top end. Then by use of the constitutive equations of the column that allows for compressibility of the column axis and non zero shear angle, see [23], [8], [28], the optimization procedure will lead to a finite value of the cross-section at the top of the column. The obtained equilibrium equations and the ones of the classical tallest column problem are homotopic. Namely, decreasing the parameters describing extensional and shear rigidity as well as the end load the solution of the problem posed here leads to the classical solution i.e. the shape of the tallest unloaded column under selfweight.

As stated by [20] as well as [17], the history of the tallest column problem includes works of Leonardo da Vinci, Galileo Galilei and Leonhard Euler who started the study of the tallest column in 1757. Indeed, problems of shape optimization for elastic columns/rods have a long history, see [41], [33], [9], [14], or [16], [37], [5] and the references therein. Besides the classical Bernoulli-Euler elastica the list can be enlarged with other types of constitutive axioms. For example optimal columns in a sense of the generalized elastica with shear and axial strain were analyzed in [38], [11] and [40] while the Eringen nonlocal theory was used in [22] where the optimal shape of the Pflüger micro/nano beam was presented. It should be noted that these generalizations of the classical Bernoulli-Euler axiom lead to non-vanishing cross-section of the optimal shapes. Therefore, in order to regularize the cross-section of the tallest column we are going to include more physical parameters in the rod/column model.

The current paper is arranged as follows. The differential equations describing equilibrium of the heavy vertical compressed column of arbitrary cross-section, with shear and axial strain, will be derived in Section 2. Next, the corresponding eigenvalue problem yielding the critical load for the prismatic column with constant cross-section and finite values of shear and extensional rigidity will be solved by use of the Goodman numerical method. In Section 4 we present the new formulation of the tallest column problem as the optimal control problem with a free boundary and use the advantages of the Pontryagin maximum principle as well as the corresponding first integrals. The necessary conditions of optimality will be derived and the heights of the uniform and the optimally shaped column will be compared. In doing so a special attention is paid to the deformation of the column axis. In Section 5, we show a possible regularization of the classical tallest column problem. Numerical results for several values of the load and column parameters are shown in Section 6. The critical load and the corresponding numerical solutions of the optimal control problems will be presented. Finally, we comment on the obtained results, compare the solution with the classical one, and give a remark on an useful inner space problem that can attract practicing engineers as well as designers of high building structures.

2 The model

Consider a heavy, vertical, sharable and compressible naturally straight column of length L clamped at the bottom point O, and free at the top end. Define a rectangular Cartesian coordinate system xOy whose x axis coincides with the column axis in the virginal state and y axis that is perpendicular to x axis. The system under consideration with some details is sketched in Fig. 1.

Let S and s respectively be the arc lengths of the column axis in the undeformed and deformed state respectively, measured from the end point O. We assume that the column has a variable cross-section of area $A = A(S)$, $S \in [0, L]$. As in [27], at the free end the column is loaded by a concentrated force of constant intensity $P \geq 0$ acting parallel to the x axis in opposite direction. For the analysis of the column element in the equilibrium configuration

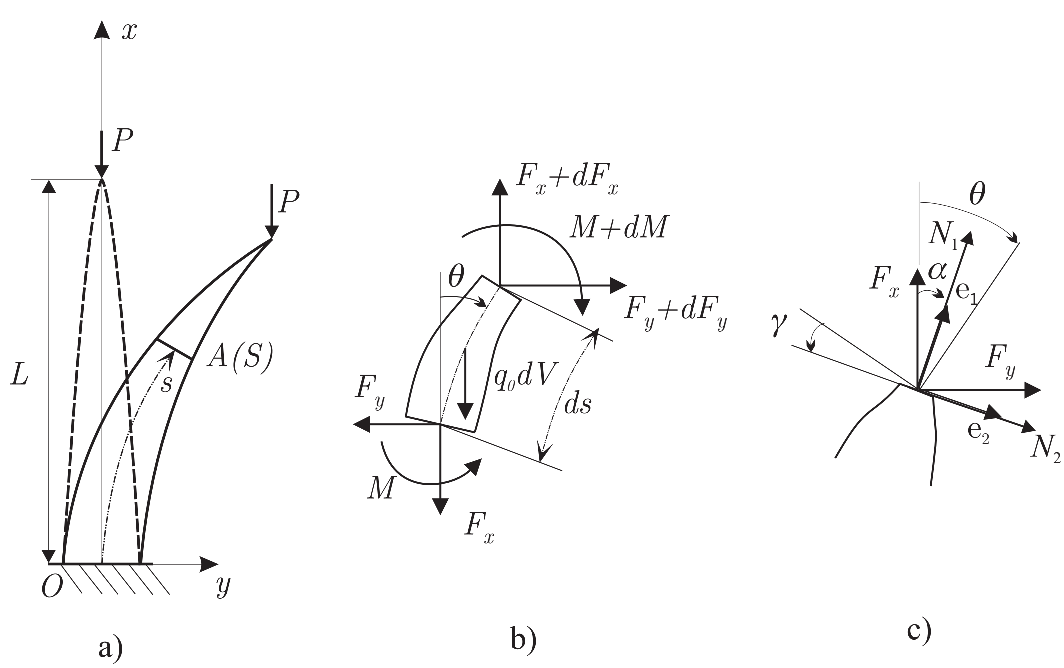

Fig. 1. Coordinate system and load configuration

we introduce the components of the internal force at arbitrary cross-section along x and y axes, say F_x and F_y respectively, together with the contact couple M and the specific weight of the column material in the undeformed state q_0, see Fig. 1b. Regarding the geometrical description of this element in the xOy coordinate system we introduce the axial strain ε, i.e.

$$\varepsilon = \frac{ds - dS}{dS},$$

the angle between the tangent to the column axis and the x axis, say θ, and the shear angle γ, i.e. the angle between the rotated (sheared, "convected") cross-section and the direction of the normal to the column axis in the deformed state, as shown in Fig. 1c. In order to relate internal shearing forces and shear angle to the introduced measures of deformation we choose Haringx's type of the internal forces decomposition, see [8] and [28]. Let $\mathbf{e}_1$, $\mathbf{e}_2$ respectively be the unit vector normal to the cross-section and the unit vector lying in the cross-section. The rotation angle of the cross-section α, as the angle between $\mathbf{e}_2$ and Oy axis, reads $\alpha = \theta - \gamma$.

In the analysis that follows we are going to compare the optimal (tallest) column with a uniform column of the same volume, say $V_u = A_u L_u$, where $A_u = const.$ and L_u denote the cross-section and the length of the uniform column respectively. Finally, we introduce the volume of the column

$$V = \int_0^L A(S)\, dS,$$

that should meet an isoperimetric constraint

$$V = V_u, \tag{1}$$

for some prescribed volume V_u, corresponding to the uniform column.

Next we follow the standard procedure of the mathematical theory of elastic rods. First, from Fig. 1b we write the geometrical relations

$$\frac{dx}{dS} = (1+\varepsilon)\cos\theta,\ \frac{dy}{dS} = (1+\varepsilon)\sin\theta, \tag{2}$$

and the corresponding differential equations describing the equilibrium of the column element of length ds

$$\begin{aligned} \frac{dF_x}{dS} &= q_0 A,\ \frac{dF_y}{dS} = 0, \\ \frac{dM}{dS} &= -F_y(1+\varepsilon)\cos\theta + F_x(1+\varepsilon)\sin\theta. \end{aligned} \tag{3}$$

The boundary conditions corresponding to (2) and (3) read

$$\begin{aligned} &x(0) = 0,\ y(0) = 0,\ \alpha(0) = 0, \\ &F_x(L) = -P,\ F_y(L) = 0,\ M(L) = 0. \end{aligned} \tag{4}$$

Secondly, in order to relate the physical and geometrical quantities introduced we follow the arguments of [30] and [23]. Namely, for strain measures $d\alpha/dS$, and Γ_1, Γ_2 defined by

$$\frac{d\mathbf{r}}{ds} = (1+\Gamma_1)\mathbf{e}_1 + \Gamma_2\mathbf{e}_2,$$

where $\mathbf{r}$ represents the position vector of an arbitrary point on the column axis in the deformed state, it follows that

$$\begin{aligned} \Gamma_1 &= (1+\varepsilon)\cos(\theta-\alpha)-1, \\ \Gamma_2 &= (1+\varepsilon)\sin(\theta-\alpha). \end{aligned} \tag{5}$$

Denoting the components of the contact force along the unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$, by N_1 and N_2 respectively, as shown in Fig. 1c, the constitutive equations/axioms of the column read

$$N_1 = EA\Gamma_1,\ N_2 = \frac{GA}{k}\Gamma_2,\ M = EI\frac{d\alpha}{dS}, \tag{6}$$

where E is the modulus of elasticity, G is the shear modulus, I is the second moment of inertia of the cross-section and k is the shear correction factor that depends on the geometry of the cross section and on the material, see [34]. For linearly elastic isotropic materials the following relation holds $E = 2(1+\nu)G$ with ν being the Poisson's ratio. The products EA and GA are also known as extensional and shear rigidity respectively. The constitutive equations (6) were recently reexamined in [28] for problems dealing with cantilever beams under different end loads. As the next step we refer to Fig. 1c again, and get the following expressions

$$\begin{aligned} N_1 &= F_x\cos\alpha + F_y\sin\alpha, \\ N_2 &= -F_x\sin\alpha + F_y\cos\alpha, \end{aligned} \tag{7}$$

representing Haringx's type of the internal forces decomposition in an arbitrary cross section of the column.

For the time being, we can omit x and y from the analysis and proceed by integrating $(3)_2$ with $(4)_5$ to conclude that $F_y = 0$ for $S \in [0,L]$ so eqs. (7) and $(3)_3$ can be simplified. Next step is to eliminate ε and θ from $(3)_3$ by the use of (5), (6) and (7). Namely, after some algebra, we get the following two-point boundary value problem describing the equilibrium configuration of the column under consideration

$$\frac{dF_x}{dS} = q_0 A,\ \frac{d\alpha}{dS} = \frac{M}{EI},$$

$$\frac{dM}{dS} = F_x\left(\frac{F_x}{EA} - \frac{kF_x}{GA} + \frac{1}{\cos\alpha}\right)\sin\alpha\cos\alpha, \tag{8}$$

$$F_x = -P,\ \alpha(0) = 0,\ M(L) = 0. \tag{9}$$

The corresponding linearized problem reads

$$\frac{dF_x}{dS} = q_0 A,\ \frac{d\alpha}{dS} = \frac{M}{EI},$$

$$\frac{dM}{dS} = F_x\left(\frac{F_x}{EA} - \frac{kF_x}{GA} + 1\right)\alpha, \tag{10}$$

and is to be solved with (9). The bending rigidity EI in (10) and the cross-section of the column A are related, i.e,

$$EI = E\psi A^2,$$

where ψ is a constant (for a circular cross-section $\psi = 1/(4\pi)$). Note that for infinite values of extensional and shear rigidity, i.e. $EA \to \infty$, $GA \to \infty$ we get the classical case corresponding to the Bernoulli-Euler bending theory.

In order to get the vertical load at the arbitrary cross section we integrate $(8)_1$ with $(9)_1$ yielding

$$F_x(S) = -P - q_0 \int_S^L A(S)\,dS, \tag{11}$$

and further

$$F_x(0) = -P - q_0 V_u, \tag{12}$$

where we used the isoperimetric constraint (1). Namely by use of (11) due to the homogeneity of the column $q_0 = const.$, the isoperimetric condition can be expressed in terms of either volume (1) or the vertical force at the bottom (12).

Next we introduce the force unit (weight of the uniform column) $F = q_0 A_u L_u$ and the following dimensionless quantities

$$t = \frac{S}{L_u},\ a = \frac{A}{A_u},\ f = \frac{F_x}{F},\ \kappa = \frac{P}{F},$$

$$m = \frac{ML_u}{E\psi A_u^2},\ \beta = k\frac{F}{GA_u},\ \mu = \frac{F}{EA_u},$$

$$\lambda = \frac{q_0 L_u^4}{E\psi V_u},\ h = \frac{L}{L_u},\ \tau = \frac{s}{L_u}. \tag{13}$$

The values $\beta = 0$ and $\mu = 0$ correspond to the classical Bernoulli-Euler elastica. Having in mind that $\beta = 2k(1+\nu)\mu$ and assuming $\nu > 0$ and $k > 1$, (for a circular cross-section $k = 1.11$), in the following we shall assume that $\beta > \mu$. By use of (13) we define $(\cdot)^{\cdot} = d(\cdot)/dt$ for $t \in [0,h]$ so the dimensionless form of boundary value problem (10), (9) reads

$$\dot{f} = a, \quad \dot{\alpha} = \frac{m}{a^2}, \quad \dot{m} = \lambda\alpha\left(f + (\mu - \beta)\frac{f^2}{a}\right), \tag{14}$$

$$f(h) = -\kappa, \quad \alpha(0) = 0, \quad m(h) = 0. \tag{15}$$

Also, the dimensionless vertical force (11) and its value at the bottom (12) become

$$f(t) = -\kappa - \int_t^h a dt, \tag{16}$$

$$f(0) = -\kappa - 1, \tag{17}$$

where we used proposed isoperimetric condition.

Two conclusions may be drawn from the linear boundary value problem (14), (15). First: the shear and extensibility effects are of the same order as shown in $(14)_3$. There are many results of the generalized plane elastica confirming the opposite influence of these effects on the critical buckling load: increasing β the critical load λ decreases, while increasing μ the critical load λ increases, see [3] and the references therein. Secondly, for any λ the problem (14), (15) has a trivial solution say f given by (16), $\alpha = 0$, $m = 0$, corresponding to the state in which the column axis remains straight. In order to determine the stability boundary of the column for given κ, β and μ we intend to determine the critical load parameter λ for which the boundary value problem (14), (15) has more then one solution. Therefore, as a preparatory result for the tallest column problem comprising shear and axial strain, we turn to the critical load problem for the uniform heavy compressed column with constitutive equations (6).

3 The critical load of the uniform column

Recall that the uniform column corresponds to $A(S) = A_u = const.$, for $S \in [0, L]$, with $L = L_u$ and in turn, $a = a(t) = 1$, for $t \in [0, h]$ with $h = 1$, see (13). In order to determine λ ensuring the nontrivial solution of $(14)_{2,3}$, $(15)_{2,3}$ for $a = 1$, $h = 1$, and for given values of β, μ and κ, first we use (16) to find

$$f(t) = -\kappa - (1 - t). \tag{18}$$

Introducing a new independent variable $\xi = 1 - t$, we get $f = -(\kappa + \xi)$. Then, denoting the derivatives with respect to ξ by prime from $(14)_{2,3}$, $(15)_{2,3}$ we get the eigenvalue problem to be solved

$$\alpha'' + \lambda(\kappa + \xi)(1 - (\mu - \beta)(\kappa + \xi))\alpha = 0,$$
$$\alpha'(0) = 0,\ \alpha(1) = 0. \tag{19}$$

In the special case when $\beta = \mu = 0$, $\kappa = 0$ the eigenvalue reads $\lambda_{cr} = 7.83735$, and is obtained as a zero of the Bessel function $J_{-1/3}(\cdot)$, see [3], while for $\beta = \mu = 0$, $\kappa > 0$ the solution of the corresponding eigenvalue problem is given in terms of Airy functions while λ_{cr} is obtained as a zero of a transcendental equation comprising several Bessel's functions and/or modified Bessel functions, see [42]. Since the eigenvalue problem (19) is more complex we conclude that the advantages of closed form solution of $(19)_1$, will be lessened by the need to fulfil $(19)_2$ and $(19)_3$. Therefore, as suggested by [29], we shall give the original boundary value problem a numerical treatment *ab initio*. The strategy suggested by Kovari was recognized in [7] and [8]. The eigenvalue problem of [7] comprises $\beta > 0$, $\mu > 0$, $\kappa > 0$ and is very similar to (19), but does not fit since we deal here with Haringx's type of the internal forces decomposition in an arbitrary cross section (or Haringx/Reissner hypothesis as stated in [28]), while in [7] Engesser's type of decomposition (or Engesser hypothesis, see [28], i.e. in normal and shear directions) was applied. In [8] where a heavy vertical column was treated, that is $\beta > 0$, $\mu > 0$, $\kappa = 0$, the eigenvalue problems for both Haringx's and Engesser's approach were derived. It was shown that the critical load predicted by Haringx's approach differs from the one predicted by Engesser's approach. Therefore the critical load in Table 2 of [8] corresponding to $\beta > 0$, $\mu > 0$, $\kappa = 0$, obtained by series solution does apply and can be used for a comparison with the numerical method suggested in [24]. The latter we intend to apply here, for more general case $\beta > 0$, $\mu > 0$, $\kappa > 0$.

The Goodman method is based on the Bliss algorithm and is as follows. First, we introduce $\eta_1 = \alpha$ and $\eta_2 = \alpha'$ and rewrite (19) in the following form

$$\eta_1' = \eta_2,\ \eta_2' = -\lambda\,(k+\xi)\,(1-(\mu-\beta)\,(\kappa+\xi))\,\eta_1,$$
$$\eta_2\,(0) = 0,\ \eta_1\,(1) = 0. \tag{20}$$

Since the eigenfunctions are known to within a multiplicative constant it is permissible to choose $\eta_1\,(0) = 1$ and an initial guess for λ, say λ^*. Integrating (20) as an initial value problem one gets the solution $\eta_i^*\,(t)$, $i = 1,2$ with $\eta_1^*\,(1)$ probably different from zero. To obtain the correct solution the value $\eta_1^*\,(1)$ must be as small as possible. The adjustment procedure requires variations defined by

$$\delta\eta_i = \eta_i - \eta_i^*,\ (i = 1,2),\ \delta\lambda = \lambda - \lambda^*,$$

to be substituted in (20) there result to a first approximation

$$\delta\eta_1' = \delta\eta_2,$$
$$\delta\eta_2' = -\lambda^*\,(k+\xi)\,(1-(\mu-\beta)\,(\kappa+\xi))\,\delta\eta_1 -$$
$$\delta\lambda\,(k+\xi)\,(1-(\mu-\beta)\,(\kappa+\xi))\,\eta_1^*, \tag{21}$$

representing the equations of differential corrections. With (21) we associate the adjoint system

$$\chi_1' = \lambda^*\,(k+\xi)\,(1-(\mu-\beta)\,(\kappa+\xi))\,\chi_2,$$
$$\chi_2' = -\chi, \tag{22}$$

where $\delta\eta_i$ and χ_i, $i = 1,2$, are related by the Bliss algorithm i.e. the one dimensional form of the Green theorem as

$$\sum_{i=1}^{2}\left[\chi_i\,(1)\,\delta\eta_i\,(1) - \chi_i\,(0)\,\delta\eta_i\,(0)\right] = -\delta\lambda\times$$

$$\int_0^1 (k+\xi)\,(1-(\mu-\beta)\,(\kappa+\xi))\,\eta_1^*(\xi)\,\chi_2(\xi)\,d\xi. \tag{23}$$

Noting that $\delta\eta_i(0)\,, i=1,2$ are zero because of the specified and the chosen initial condition as well as that $\delta\eta_1(1)=\eta_1(1)-\eta_1^*(1)=-\eta_1^*(1)$ we pick $\chi_1(1)=1$ and $\chi_2(1)=0$ in order to solve (22) by backward integration so from (23) the increment that will adjust λ^* reads

$$\delta\lambda = \frac{\eta_1^*(1)}{\int_0^1 (k+\xi)\,(1-(\mu-\beta)\,(\kappa+\xi))\,\eta_1^*(\xi)\,\chi_2(\xi)\,d\xi}.$$

The procedure is then repeated for $\lambda^*=\lambda+\delta\lambda$ until a declared convergence is achieved. As stated by Goodman the method is truly equivalent to the Newton method of finding roots of transcendental equations so there is no guarantee that the convergence will occur. However if λ^* is chosen close to the eigenvalue λ_{cr} this iterative method is very efficient.

With this preparation done we may turn to the problem how can we increase the height of the column by the material redistribution along its longitudinal axis.

4 The design of the tallest column

In order to change the cross-section of the column that will increase its height with respect to the uniform one of the same volume/weight that will buckle for the same critical load we go back to $(2)_1$, $(5)_1$, $(6)_1$, and $(7)_1$. Namely, after linearization of $(2)_1$, $(5)_1$ and $(7)_1$ we get $dx=ds$, $\Gamma_1=\varepsilon$ and $N_1=F_x$ respectively. Then $(6)_1$ becomes $F_x=EA\varepsilon$ and further, by use of (13), we get $\varepsilon=f\mu/a$, and

$$d\tau = \left(1+\frac{f\mu}{a}\right)dt. \tag{24}$$

In case of $\mu>0$ the uniform column of cross-section $a=1$ and the initial length $h=1$ on the trivial solution is of length $h_u^0<h$ since

$$h_u^0 = \int_0^h 1\cdot d\tau = 1-\kappa\mu-\frac{\mu}{2}, \tag{25}$$

where we used (24) and (18).

Given $\beta>\mu>0$ and $\kappa>0$ we determined λ_{cr} for the uniform column corresponding to $a=1$ and $h=1$. Having in mind that the prescribed volume i.e. the isoperimetric condition was taken into account by prescribing the vertical force at the bottom (17), we pose the following optimal control problem: given $\beta>\mu>0$, $\kappa>0$ and λ_{cr} corresponding to the critical load of the uniform column, find $a(t)>0$ that will maximize the dimensionless height of the column here denoted by h. By the analogy to the brachistochrone problem or the simplest time optimal problem as declared in [2], we propose to choose the cross-sectional area $a(t)$, $t\in[0,h]$ such that the height of the column will attain its maximal value, that is

$$\max_{a\in\mathscr{U}} h = \int_0^h 1\cdot d\tau = \int_0^h (1+\varepsilon)\,dt, \tag{26}$$

where we used (24), subject to

$$\dot{f} = a,\ \dot{\alpha} = \frac{m}{a^2},\ \dot{m} = \lambda_{cr}\alpha\left(f + (\mu - \beta)\frac{f^2}{a}\right), \tag{27}$$

and

$$f(0) = -\kappa - 1,\ \alpha(0) = 0,\ f(h) = -\kappa,\ m(h) = 0. \tag{28}$$

In (27) it was assumed that a belongs to the set of admissible cross-sectional areas, defined as

$$\mathscr{U} = \{a:\ a \in C(0,h),\ a \geq 0\}.$$

Note that $(28)_1$, $(28)_3$ correspond to the isoperimetric constraint. Also note that the free end in the optimality criteria (26) is not specified so instead of ordinary (Lagrangian) variations we shall use generalized variations that will be denoted by Δ.

Introducing the adjoint variables p_f, p_α and p_m, the Pontryagin function (aka Hamiltonian) reads

$$H = 1 + \frac{f\mu}{a} + p_f a + p_\alpha \frac{m}{a^2} + p_m \lambda_{cr}\alpha\left(f + (\mu - \beta)\frac{f^2}{a}\right). \tag{29}$$

Selecting a to be the control variable and applying the Pontryagin maximum principle in its standard form, as in [26] or [2], we write the co-adjoint equations, say $\dot{p}_\varphi = -\partial H/\partial\varphi$ for $\varphi = f, \alpha, m$, the transversality conditions for generalized variations, and the optimality condition $\partial H/\partial a = 0$ respectively as follows

$$\dot{p}_f = -\frac{\mu}{a} - p_m\lambda_{cr}\alpha\left(1 + 2(\mu - \beta)\frac{f}{a}\right),$$

$$\dot{p}_\alpha = -p_m\lambda_{cr}\left(f + (\mu - \beta)\frac{f^2}{a}\right),\ \dot{p}_m = -\frac{p_\alpha}{a^2}, \tag{30}$$

$$\left[-p_f\Delta f - p_\alpha\Delta\alpha - p_m\Delta m + H\Delta h\right]\Big|_0^h = 0, \tag{31}$$

$$-\frac{f\mu}{a^2} + p_f - 2p_\alpha\frac{m}{a^3} - \lambda_{cr}(\mu - \beta)\,p_m\alpha\frac{f^2}{a^2} = 0. \tag{32}$$

The necessary condition ensuring the maximum of H reads

$$\frac{\partial^2 H}{\partial a^2} = 2\frac{f\mu}{a^3} + 6p_\alpha\frac{m}{a^4} + 2\lambda_{cr}(\mu - \beta)\,p_m\alpha\frac{f^2}{a^3} < 0. \tag{33}$$

Taking into account prescribed boundary conditions (28) with $\Delta h \neq 0$ from (31) we get the following transversality conditions

$$p_\alpha(h) = 0,\ p_m(0) = 0,\ H(h) = 0. \tag{34}$$

Since f is specified at both ends, due to the imposed isoperimetric condition, the corresponding adjoint variable p_f is not specified at the boundaries of the interval $[0,h]$. A re-examination of $(27)_{2,3}$ and $(30)_{2,3}$ as well as the corresponding boundary conditions $(28)_{3,4}$ and $(34)_{1,2}$ leads to possible connections between the state variables α and m and the costate variables p_α and p_m. Thus, in order to fulfill (33) we pick

$$p_\alpha = -m,\ p_m = \alpha, \tag{35}$$

so the adjoint variables p_α and p_m can be omitted from the further analysis. Besides the condition (33) now becomes

$$\frac{\partial^2 H}{\partial a^2} = 2\frac{f\mu}{a^3} - 6\frac{m^2}{a^4} + 2\lambda_{cr}(\mu-\beta)\alpha^2\frac{f^2}{a^3} < 0,$$

and is satisfied since $f < 0$, $\mu - \beta < 0$, and $a > 0$. Therefore the necessary condition for $\max_{a\in\mathscr{U}} H$, and in turn, according to the Pontryagin maximum principle, $\max_{a\in\mathscr{U}} h$, that is $\partial^2 H/\partial a^2 < 0$ is satisfied. We note that the sufficient condition for $\max_{a\in\mathscr{U}} H$ is more complicated, see [31], [12], and will not be analyzed here.

With (35) the Pontryagin function (29) and the corresponding optimality condition (32) become

$$H = 1 + \frac{f\mu}{a} + p_f a - \frac{m^2}{a^2} + \lambda_{cr}\alpha^2\left(f + (\mu-\beta)\frac{f^2}{a}\right),$$

$$\frac{\partial H}{\partial a} = -\frac{f\mu}{a^2} + p_f + 2\frac{m^2}{a^3} - \lambda_{cr}(\mu-\beta)\alpha^2\frac{f^2}{a^2} = 0. \tag{36}$$

Next we take the advantage of the optimization formulation (26) as follows. As in [4], we note that the Pontryagin function H does not depend on the independent variable explicitly, here the dimensionless arc length t, i.e.,

$$\frac{\partial H}{\partial t} = 0,$$

so it represents the first integral $H = const.$ that is equal to zero, due to the transversality condition corresponding to the free end of the optimization problem $(34)_3$. Therefore, $H = 0$, on $[0,h]$, so

$$1 + \frac{f\mu}{a} + p_f a - \frac{m^2}{a^2} + \lambda_{cr}\alpha^2\left(f + (\mu-\beta)\frac{f^2}{a}\right) = 0.$$

Finally, the adjoint variable p_f can also be eliminated from further analysis by use of two first integrals $\partial H/\partial a = 0$ and $H = 0$ given by (36) and (37) respectively. This is the main advantage of this novel formulation. Namely, multiplying (36) by $-a$ and adding the result to (37) after some algebra will reduce the optimality condition to the following quadratic equation

$$a^2 - \frac{2\left(\lambda_{cr}(\beta-\mu) f^2\alpha^2 - f\mu\right)}{1+\lambda_{cr} f\alpha^2} a - \frac{3m^2}{1+\lambda_{cr} f\alpha^2} = 0.$$

We make two remarks here. Firstly, all adjoint variables are eliminated from the optimal control problem. Namely, the connections (26) are obvious, while the first integrals $H = 0$ and $\partial H/\partial a = 0$ that follow form the chosen optimality criteria with free end point (26) and the isoperimetric condition given in terms of vertical load $(28)_1$, $(28)_3$ lead to the elimination of the third one. Second, the quadratic equation determining the optimal cross section of the tallest column (37) is very tractable since both free term and the coefficient of the linear term are negative $(f < 0)$ so its solution is easy to find.

The optimal cross-section $\hat{a} = \hat{a}(t)$ on $[0,h]$ reads

$$\hat{a} = \frac{\lambda_{cr}(\beta-\mu) f^2\alpha^2 - f\mu}{1+\lambda_{cr} f\alpha^2} + \sqrt{\left[\frac{(\lambda_{cr}(\beta-\mu) f^2\alpha^2 - f\mu)}{1+\lambda_{cr} f\alpha^2}\right]^2 + \frac{3m^2}{1+\lambda_{cr} f\alpha^2}}. \tag{37}$$

Substituting the boundary conditions $(28)_{2,4}$ into (37) respectively yields the values of the optimal cross-section at the boundaries $t = 0$ and $t = h$, i.e.,

$$(1+\kappa)\mu + \sqrt{[(1+\kappa)\mu]^2 + 3m(0)^2} \geq \hat{a}(t) \geq 2\frac{\lambda_{cr}(\beta-\mu)(\kappa)^2\alpha^2 + \kappa\mu}{1+\lambda_{cr}\kappa\alpha^2} > 0.$$

Thus for $\kappa > 0$ the non-vanishing cross-section is ensured. In order to find the maximal height of the column we have to scale two-point boundary value problem (27), (28), with (37). Namely, formally adding $\dot{h} = 0$ to (27), introducing the new independent variable $\zeta = t/h$, $\zeta \in [0,1]$ and denoting the derivatives with respect to ζ again by dot, the maximal height of the column h is determined by the solution of the following two-point boundary value problem

$$\dot{f} = h\hat{a},\ \dot{\alpha} = \frac{hm}{\hat{a}^2},$$

$$\dot{m} = h\lambda_{cr}\alpha\left(f + (\mu-\beta)\frac{f^2}{\hat{a}}\right),\ \dot{h} = 0, \tag{38}$$

subject to

$$f(0) = -\kappa - 1,\ \alpha(0) = 0,\ f(1) = -\kappa,\ m(1) = 0. \tag{39}$$

In order to solve two-point boundary value problem (38), (39) the classical shooting technique is applied. We stress that the solution to (38),(39) determines the height of the tallest column h. The shape of the tallest column $a_{tallest}(t)$, $t \in [0,h]$ is determined form the solution of (27),(28) and $a = \hat{a}$ given by (37). In case of $\mu > 0$ the height of the optimal/tallest column on the trivial solution reads

$$h^0 = \int_0^h \left(1 + \frac{f(t)\mu}{a_{tallest}(t)}\right) dt. \tag{40}$$

5 The regularization of the classical case

In order to obtain the Keller-Niordson solution that is $h_{tallest} = 2.034$ for $\lambda_{cr} = 7.833$ and $\kappa \to 0$, $\beta = \mu = 0$ we start with (26) that is reduced to

$$\max_{a \in \mathscr{U}} h = \int_0^h 1 \cdot dt,$$

and (27) that are simplified to

$$\dot{f} = a, \quad \dot{\alpha} = \frac{m}{a^2}, \quad \dot{m} = \lambda_{cr} \alpha f, \tag{41}$$

subject to (28). The equations corresponding to (37), $(36)_2$, $(30)_1$ respectively read

$$1 + pa - \frac{m^2}{a^2} + \lambda_{cr} f \alpha^2 = 0,$$

$$p + 2\frac{m^2}{a^3} = 0,\ \dot{p} = -\lambda_{cr}\alpha^2, \tag{42}$$

where we used $(35)_2$ and where we put $p \equiv p_f$. The optimality condition corresponding to (37) for $\beta = \mu = 0$ reads

$$a = \frac{\sqrt{3}m}{\sqrt{1 + \lambda_{cr} f \alpha^2}}. \tag{43}$$

Note that the boundary condition $(28)_4$ causes the singularity of (41), since $a(h) = 0$ and the presence of the trivial solution $\alpha = m = 0$ lead to wayward numerics. Thus, in order to make the problem more tractable, motivated by the arguments related to the classical problem presented in [4], we shall use the first integrals of the above system as follows.

Multiplying $(41)_2$ by m and adding the result to $(41)_3$ multiplied by α one gets

$$(m\alpha)^{\cdot} = 4\frac{m^2}{a^2} - 1, \tag{44}$$

where we used (42). Integrating (44) yields

$$\int_0^h \frac{m^2(\xi)}{a^2(\xi)} d\xi = \frac{h}{4}. \tag{45}$$

Similarly, multiplying $(41)_1$ by p and adding the result to $(42)_3$ previously multiplied by f lead to

$$(pf)^{\cdot} = 1 - 5\frac{m^2}{a^2}, \tag{46}$$

where as before we used Hamiltonian $(42)_1$ and the necessary condition for optimality $(42)_2$. By integration of (46) on $[0,h]$ one gets

$$-p(h)\kappa + p(0)(1+\kappa) = -\frac{h}{4}, \tag{47}$$

where we used $(28)_{1,3}$ and (45). Assuming the finite value of $p(h)$, in the limiting case $\kappa \to 0$ from (47) it holds

$$p(0) = -\frac{h}{4}. \tag{48}$$

Since $a(0) = \sqrt{3}m(0)$, see (43), substituting this and (48) in $(42)_2$ one finds

$$m(0) = \frac{8}{3\sqrt{3}h}, \; a(0) = \frac{8}{3h}. \tag{49}$$

On the other hand eliminating m^2/a^2 from (44) by use of $(42)_{1,2}$ leads to

$$(m\alpha)^{\cdot} = -2pa - 1,$$

while multiplying $(41)_3$ by m yields

$$\left(m^2\right)^{\cdot} = 2\lambda_{cr} m\alpha f.$$

With this preparation and new variables $\rho = m\alpha$ and $\omega = m^2$ the problem turns to

$$\dot{f} = a, \; \dot{\rho} = -2pa - 1, \; \dot{\omega} = 2\lambda_{cr} f\rho,$$
$$\dot{p} = \frac{2+3pa}{2f}, \; \dot{h} = 0.$$

where we expressed $\lambda_{cr}\alpha^2$ from $(42)_1$, $(42)_3$ and use $(42)_2$ to get $(50)_4$. The boundary conditions correspond to (50) read

$$f(0) = -\kappa - 1, \; \rho(0) = 0, \; f(h) = -\kappa,$$

$$\omega(h) = 0, \; p(h)\kappa - p(0)(1+\kappa) = \frac{h}{4}. \tag{50}$$

The system (50), (50) should be complemented by the optimality condition $(42)_2$ that is

$$a = \sqrt[3]{\frac{-2\omega}{p}}. \tag{51}$$

Note that there is no singularity in (50) since $-\kappa - 1 < f < -\kappa$, $(\kappa \to 0)$, since $\omega(h) = 0$ and in turn $a(h) = 0$ causes no problem anymore. Besides, the infinite grow of α at $t = h$ does not influence the problem. The upper boundary of the parameter κ should be chosen to keep the same value of the critical load λ_{cr}.

The last step is to avoid integration over the unspecified interval $[0,h]$ and perform one on $[0,1]$, so once again we use $\zeta = t/h$, $\zeta \in [0,1]$ and keep dot to denote the derivative with respect to ζ, so the two point boundary value problem that covers h of the classical solution reads

$$\dot{f} = h\sqrt[3]{\frac{-2\omega}{p}},\ \dot{\rho} = -2h\sqrt[3]{-2\omega p^2} - h,\ \dot{\omega} = 2\lambda_{cr} h f \rho,$$

$$\dot{p} = h\frac{2 + 3\sqrt[3]{-2\omega p^2}}{2f},\ \dot{h} = 0, \tag{52}$$

$$f(0) = -\kappa - 1,\ \rho(0) = 0,\ f(1) = -\kappa,$$

$$\omega(1) = 0,\ p(1)\kappa - p(0)(1+\kappa) = \frac{h}{4}. \tag{53}$$

6 Results and discussion

In this section, in Table 1 and in Figures 2 and 3 we present numerical solutions of the eigenvalue problem (19) and two point boundary value problems (38), (39) and (52), (53). Then we comment on the obtained results.

In Table 1, in the fourth column, we present the eigenvalue λ_{cr} corresponding to (19) obtained by Goodman's method for several values of load and column parameters κ, β and μ. In the fifth column the length of the uniform column h_u^0 in its trivial configuration obtained by (25) is shown. In case of $\mu > 0$ this value is less then 1. In the sixth column we present the height of the tallest column h as the solution of (38), (39). The corresponding height of the tallest column in its trivial configuration h^0 obtained by (40) and the base cross-section $\hat{a}(0)$ obtained from (37) for $t = 0$ are given in the 7th and 8th column respectively. In doing so, in case of $\mu > 0$ we solve (38), (39) to find h and then (27),(28) to find the optimal shape from (37). Note that both uniform and the optimally shaped column i.e. the tallest column, are of the same weight and will sustain the same critical load λ_{cr} without buckling. However the latter is much higher. In case when $\beta = \mu = 0$ we solve (52), (53) for the corresponding λ_{cr} and κ and present the solutions in the first and the third row. A tiny number within the following table ι stands for 1×10^{-8}.

TABLE 1. Results for critical load and corresponding tallest column.

κ	β	μ	λ_{cr}	h_u^0	h	h^0	$\hat{a}(0)$
ι	0	0	7.83735	1	2.0206	2.0206	1.315
ι	3ι	ι	7.83735	1	2.0205	2.0205	1.311
0.1	0	0	6.00976	1	1.2810	1.2810	1.304
0	0.05	0	7.65587	1	2.0004	2.0004	1.310
0	0.1	0	7.48241	1	2.0176	2.0176	1.309
0	0.1	0.04	7.62056	0.98	2.0031	1.9792	1.321
0	0.03	0.01	7.76376	0.995	1.9983	1.9924	1.313
0.1	0.03	0.01	5.94643	0.994	1.2860	1.2784	1.306
0.005	0.03	0.01	7.64856	0.995	1.6304	1.6242	1.313

First we comment on the critical load λ_{cr}. For $\kappa = 0$ we cover all results presented in Table 2 of [8] obtained by Newton's method combined with corresponding series solution. Regarding the influence of finite values of shear and extensional rigidity the well known results of the generalized plane elastica theory were confirmed as expected. Namely, the finite value of shear rigidity $\beta > 0$ decreases, while while the finite value of extensional rigidity μ increases the critical load λ_{cr}. Having in mind that these effects are of the same order, or strictly coupled due to he opposite influence on the critical load, we may speculate that both effect should be taken into account in optimization problems. Finally, comparing the first and the second row we recognize the homotopy between this problem and the classical one as declared in the introduction since the solution of (38), (39) is very close to the solution of (52), (53).

Next we comment the effects of optimization. A comparison between 5th and 7th column shows that the optimally shaped column in its upwards position is higher than the prismatic constant cross-section column of the same volume that will buckle for the same value of λ_{cr}. For the parameter values presented in Table 1 the optimal columns are between 28.7 and about 100% higher with respect to the uniform column with the same volume/weight and the critical load.

In case of a shearable rod ($\beta > 0$, $\mu = 0$) from the results presented in the first, the fourth and 5th rows of Table 1 we conclude that increasing β the maximal height of the column h increases while the cross-section at the bottom end decreases. Namely, increasing β the optimal column becomes more slender since $h/\hat{a}(0)$ decreases. The critical load λ_{cr} decreases too so the shearable rod loaded with the same weight as the uniform column will become taller as the result of the optimization procedure. Introducing both effects for $\mu > 0$ and $\beta > \mu$ we conclude that the opposite is true.

In order to present the shape of the tallest column in an observable 3D space, let us recall the Pearson formulation of the Lagrange problem: to find the curve which by its revolution about an axis in its plane determines the column of greatest efficiency, see [16]. Note that the efficiency here means the tallest column of circular cross-section with respect to the uniform one of radii $r_u = \sqrt{1/\pi} = 0.56419$. In Fig. 2 we present the classical solution

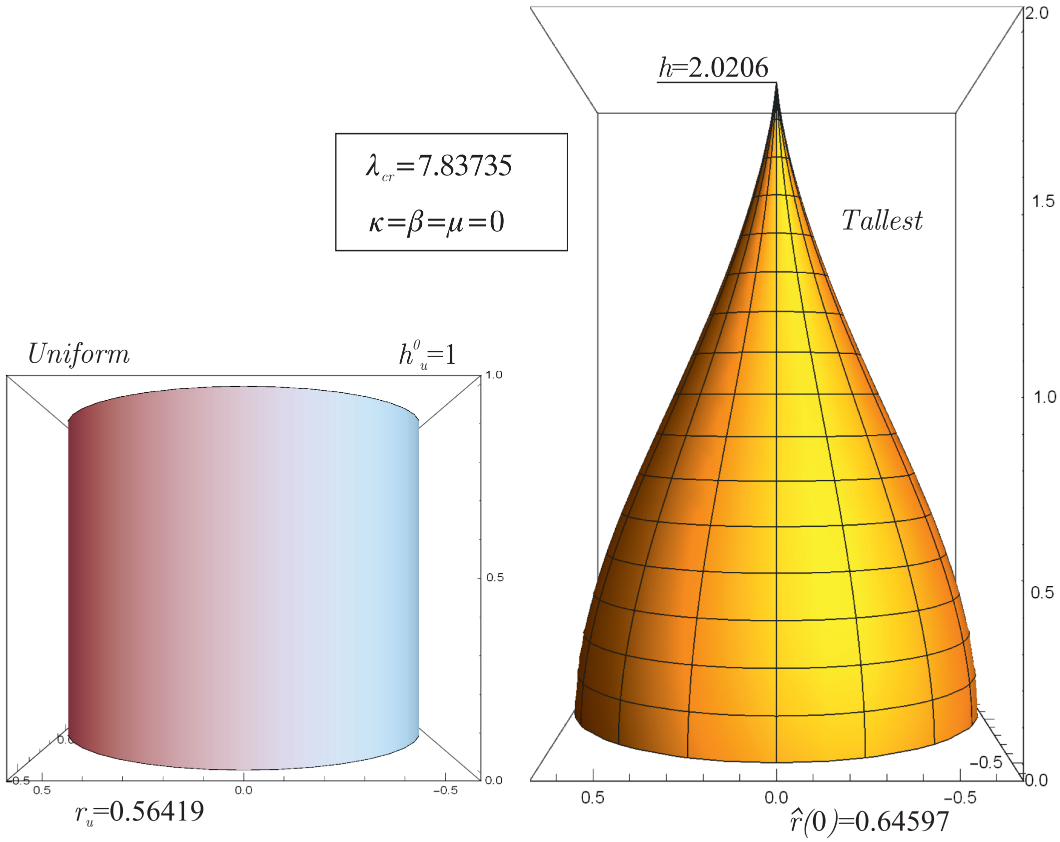

Fig. 2. The uniform and the tallest column in a sense of the Bernoulli-Euler theory

corresponding to the Bernoulli-Euler elastica theory and the first row of Table 1. Namely, calculating $\hat{r} = \sqrt{\hat{a}(t)/\pi}$ we present the uniform column of circular cross-section of radii r_u and the optimally shaped (tallest) column of the same volume that will buckle with the same critical load.

The tallest column corresponding to the last row of Table 1 is shown in Fig. 3. Note that the uniform column that was initially of length 1 was shorten to 0.995 due to axial strain.

Next we comment on the results of some other investigations on the classical problem. The solution of (52), (53) for $\lambda_{cr}^{KN} = 7.833$, $\kappa = 1 \times 10^{-7}$ yields $h = 2.0337984$, what covers the result of [27] where the height of the tallest column $h_{KN} = 2.034$ was reported. Besides, with $h = 2.0337984$ the eigenvalue of the optimally shaped column reads

$$\lambda_{opt} = \lambda_{cr}^{KN} \times h^4 = 134.02.$$

This value compares well with the results maximizing the critical load for given volume obtained earlier: $\lambda_{opt} = 134.19$, in [27], $\lambda_{opt} = 134.154$, in [32], $\lambda_{opt} = 134.1944$, in [21] and $\lambda_{opt} = 134.1935$ in [4]. Next we comment on the eigenvalue λ_{cr} presented in the above table that differs from $\lambda_{cr}^{KN} = 7.833$ used by Keller and Niordson. In case of $\beta = \mu = 0$,

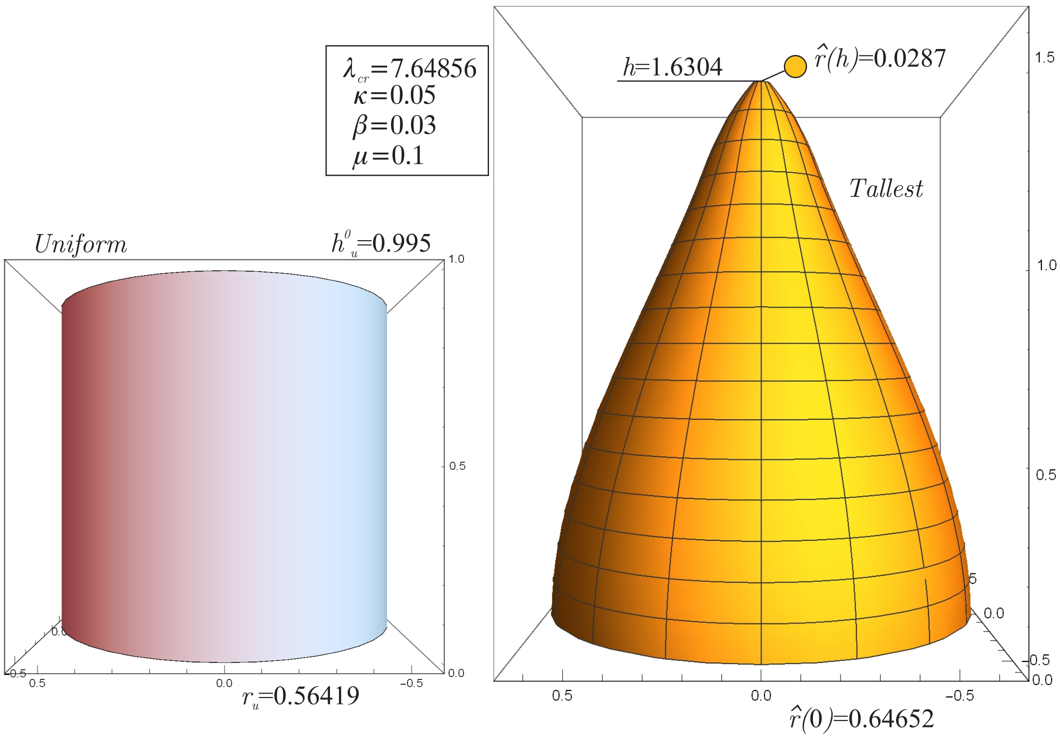

Fig. 3. The uniform and the tallest column in a sense of the generalized elastica theory

$\kappa = 0$,for our result of Table 1 we calculated $\lambda_{cr} = 7.83734744$ and round it to 7.83735. Regarding that case it is known that $\lambda_{cr} = \frac{3}{2} j^2$ where j is the smallest root of $J_{-\frac{1}{3}}(j) = 0$, as shown in [42]. Thus, Wang and Drachman reported $\lambda_{cr} = 7.83735$, that coincides with the value obtained here. It should be noted that the difference $\Delta\lambda = \lambda_{cr} - \lambda_{cr}^{KN} = 0.00435$ corresponds to the dimensionless end load $\kappa = 1.8498 \times 10^{-4}$. Since κ decreases the height of the tallest column, in the classical case h should be less then the one obtained by Keller and Niordson here denoted by h_{KN}. Regarding results recently presented in [4] where the problem was considered on $[0,1]$ we may add that the estimation of $a(0) = 8/3$ presented therein agrees with $(49)_2$ for $h = 1$. Also the estimation that

$$\frac{a(0)}{m(0)} = \frac{\sqrt{3}}{2},$$

obtained in [4] is analogue to the one obtained here

$$\frac{a(0)}{m(0)} = \sqrt{3},$$

since the difference is due to different value of the constant representing Hamiltonian.

Next we give several remarks on the optimality condition (37) and present the advantage of the optimization approach given by formulation (26). First we comment on the tallest column obtained in the sense of the classical Bernoulli-Euler elastica theory. Note that in the approach to the problem presented, for example in [4], where the minimization of the volume is used, the optimality condition leads to a two-point boundary value problem with singularity at the top of the column where its cross-section vanishes. In [4] the singularity problem was avoided by use of the new first integrals. Despite the existing singularity [18] proved the existence and uniqueness of the solution of the problem (14), (15) with $\beta = \mu = 0$, $\kappa = 0$. The singularity and the behavior of the column near the top were investigated in [21]. In [27] the singularity is treated by formulating an equivalent integral equation. The list of references related to this issue how to deal with the singularity at the top and do numerics in the presence of the trivial solution when $\mu = \beta = 0$ can be enlarged by many more [10], [16], [15], [19], to mention just a few. However, a strategy to resolve the anomaly may be a different one, that is to introduce more physical parameters in the model what was actually done here by choosing the column model that can suffer flexion, shear and axial strain. Namely, for $\beta > \mu > 0$ the same constitutive axiom as (6) was used in [40] where the optimality condition as a solution of the depressed cubic was obtained. The optimal cross section therein was obtained by use of the Chebyshev root function. Also, it was shown that imposing shear in the column model will eliminate points where the cross-section of the column vanish. The regularization of the column cross-section and in turn, avoidance of the singularities in the differential equations in optimal control problems by imposing finite values of extensional and shear rigidity was presented in [38]. The nonlinear corresponding optimality condition therein was solved numerically by the bisection method. Finally, the optimality condition such as (37) can be differentiated and coupled with (38), (39) and $(28)_1$ provided an initial condition $a(0)$ is known in advance. This was the case in [22], however it does not work here because the initial condition $a(0)$ can not be easily determined.

Regarding the physical homotopy declared in the introduction, it was shown that decreasing κ, $\beta > \mu$ and calculating the corresponding λ_{cr} we get closer to the classical solution. Namely, for $\kappa = 1 \times 10^{-8}$and $\mu = \beta/3 = 1 \times 10^{-8}$ we get the maximal height $h = 2.0205$ what agrees well with $h = 2.0206$ obtained for the classical case and the same value $\lambda_{cr} = 7.83735$. We conclude that in some sense the regular system (38), (39) and the classical system obtained within the framework of the Bernoulli-Euler elastica theory for $\kappa = \beta = \mu = 0$ are homotopic since three physical parameters κ, β and μ can transform one system to another, as can be seen by comparison of the results presented in the first and the second row of Table 1. It can be confirmed for the results presented in the third row too.

Finally, we comment on the dimensionality of the optimal control problem. The formulation in which both the critical load and the weight of the column are fixed while its height remains unspecified leads to the additional first integral $H = 0$ and in turn to elimination of all adjoint variables. This simplifies the shooting procedure as the standard technique of solving structural optimization problems.

7 Closure

In this paper we proposed the new pattern of determining the shape of the vertical column that will increase its height with respect to the uniform one that is of the same weight and will sustain the same critical load λ_{cr} without buckling. The new pattern comprises the optimal control problem with a free end point and the constitutive equations of the column that take flexure, compression and shear into account. Namely, in order to maximize the height for the fixed critical load and the weight of the column, we used the concept of generalized variations within the framework of the Pontryagin maximum principle for problems with the isoperimetric constraint. As in [27] we introduced the compressing force at the free end of the column and allow this force to vanish. Our main results are:

1. With (14), (15), we described the linearized equilibrium equations of the heavy vertical column with the end load in the presence of the shear and axial strain on unspecified domain. With (17) we incorporated the isoperimetric constraint into the problem. The parameters κ, β and μ therein correspond to the dimensionless end load, shear and extensional rigidity respectively. These equations correspond to Haringx's type of the internal force decomposition.

2. The eigenvalue problem corresponding to (14), (15), for the column of the constant cross-section and the unit length was given by (19) and solved numerically by use of Goodman's method. The critical load λ_{cr} for several values of load and column parameters κ, β and μ are given in the first four columns of Table 1, where the influence of the end load as well as the finite values of shear and extensional rigidity on the critical load was shown. Increasing κ and β decreases the critical load λ_{cr} while increasing μ increases the critical load as expected.

3. The new form of the optimality criteria was stated in (26). The application of the Pontryagin maximum principle for the corresponding Hamilton function introduced in (29) led to necessary conditions of optimality (30) - (33). The simple observations of the state and costate equations $(27)_{2,3}$ and $(30)_{2,3}$, together with the optimality condition (36) and the first integral (37) that follows from the unspecified terminal arc coordinate and form of the Hamiltonian, led to the elimination of all costate variables. The corresponding optimal cross-section of the tallest column was given in (37) as a solution of the quadratic equation (37).

4. It was shown that the cross-section of the column does not vanish for positive values of κ, β and μ. In the case when these constants tend to zero the problem and its solution cover the classical tallest column posed in the framework of the classical Bernoull-Euler elastica theory. Yet another regularization of this classical problem was given by (52), (53). The result of [27] was covered. We showed that the classical solution and the solution obtained in sense of the generalized plane elastica theory are homotopic in a physical sense. Namely, decreasing the compressive load acting

at the top of the column and parameters describing shear and extensional rigidity the solutions of the two-point boundary value problems (38), (39) and (52), (53) coincide.

5. For several values of the load and column parameters, the numerical solutions of the boundary value problem corresponding to the optimal control problem posed here (38), (39), were presented in the columns 5 to 8 of Table 1. Both maximal $h > 1$ and in case of $\mu > 0$ the height in the corresponding trivial configuration of the tallest h^0 with respect to the uniform column $h_u^0 < 1$ are significantly higher.

6. The solutions of the classical problem and the generalized problem in the observable 3D space for the column of circular cross-section were shown in figures 2 and 3 respectively.

7. The obtained numerical results quantitatively show how the finite values of shear and extensional rigidity influence the optimal shape of a heavy compressed column with and without the compressive force at the free end. It was shown that finite value of the shear rigidity increases the height of the tallest column and decreases its cross-section at the bottom making the tallest column more slender, while the finite value of the extensional rigidity decreases its height and increases its cross-section at the bottom making the tallest column more stocky.

The pattern presented here may be applied to solve some practical problems in high building design, for example the one related to the useful inner space. Namely, any flat plate of given weight, say $n \cdot \kappa$, $n \in \mathbb{N}$, can be supported by n columns like the ones presented in Fig. 3. However using the tallest (optimally shaped) instead of the uniform columns yields 67% more of the useful inner space. Knowing that the end load of each column κ decreases along the line from bottom to top of the building, one can predict net gain of the useful inner space while keeping the same cost of the material since both columns are of the same weight. Besides, the critical load is the same too, so the physical and in turn engineering parameters involved in that load will be the same too. Therefore the next step to be done is to examine if this pattern can be related more to demands and attention of practicing engineers as well as designers of high building structures. Several very useful questions on this issue can be found in [13] and [1].

References

[1] M. Aldwaik, H. Adeli, *Advances in optimization of highrise building structures*, Struct. Multidisc. Optim. 50 (2014) 899–919.

[2] V. M. Alekseev, V. M.Tikhomirov, S. V. Fomin, *Optimal control*. Plenum Publishing Corporation, New York, 1987.

[3] T. M. Atanackovic, *Stability Theory of Elastic Rods*, World Scientific, Singapore, 1997.

[4] T. M. Atanackovic, *The tallest column problem: New first integrals and estimates*, C. R. Mecanique. 347 (2019), 626–631.

[5] T. M. ATANACKOVIC, V. B. GLAVARDANOV, *Optimal shape of a heavy compressed column*, STRUCT. MULTIDSC. OPTIM. 28 (2004), 388–396.

[6] T. M. ATANACKOVIC, A. P. SEYRANIAN,, *Application of Pontryagin's principle to bimodal optimization problems*, STRUCT. MULTIDISC. OPTIM. 37 (2008), 1–12.

[7] T. M. ATANACKOVIC, D. T. SPASIC, *Effect of shear and compressibility on stability of a heavy column with an end load*, Q. JL. MECH. APPL. MATH. 44 (1991), 601–617.

[8] T. M. ATANACKOVIC, D. T. SPASIC, *A model for plane elastica with simple shear deformation pattern*, ACTA MECH. 104 (1994), 241–253.

[9] N. V. BANICHUK, A. A. BARSUK, *Design of an optimum column with elastic clamping*, STRUCT. OPTIMIZATION. 9 (1995), 254–257.

[10] D. C. BARNES, *The shape of the strongest column is arbitrarily close to the shape of the weakest column*, QUART. APPL. MATH. 4 (1988), 605–609.

[11] D. J. BRAUN, *On the optimal shape of compressed rotating rod with shear and extensibility*, INT. J. NONLIN. MECH. 43 (2008), 131–139.

[12] A. C. CHIANG, *Elements of Dynamic Optimization*, MCGRAW-HILL, NEW YORK, 1992.

[13] M. Z. COHN, *Theory and practice of structural optimization*, STRUCT. OPTIMIZATION. 7 (1994), 20–31.

[14] S. J. COX, *The shape of the ideal column*, MATH. INTELL. 14 (1992), 16–24.

[15] S. J. COX, J. B. MCCARTHY, *The Shape of the Tallest Column*, SIAM J. MATH. ANAL. 29 (1998), 547–554.

[16] S. J. COX, M. L. OVERTON, *On the optimal design of columns against buckling*, SIAM J. MATH. ANAL. 23 (1992), 287–325.

[17] J. I. DÍAZ, M. SAUVAGEOT, *Euler's tallest column revisited*, NONLINEAR ANAL. REAL WORLD APPL. 11 (2010), 2731–2747.

[18] Y. V. EGOROV, *On the tallest column*, C. R. MECANIQUE. 338 (2010), 266–270.

[19] Y. V. EGOROV, *On Euler's problem*, SB. MATH. 204 (2013), 539–562.

[20] M. V. ESPÍ, J. C. BRAVO, C. O. ROJAS, *On Galileo's tallest column*, MATH. PROBL. ENG. (2015), HTTP://DX.DOI.ORG/10.1155/2015/649341.

[21] J. FARJOUN, J. NEU, *The tallest column: a dynamical system approach using a symmetry solution*, STUD. APPL. MATH. 115 (2005), 319–337.

[22] V. B. GLAVARDANOV, D. T. SPASIC, T. M. ATANACKOVIC, *Stability and optimal shape of Pflüger micro/nano beam*, INT. J. SOLIDS STRUCT. 49 (2012), 2559–2567.

[23] Y. GOTO, T. YOSHIMITSU, M. OBATA, *Eliptic integral silutions of plane elestica with axial and shear deformations*, INT. J. SOLIDS STRUCT. 26 (1990), 375–390.

[24] T. R. GOODMAN, *The numerical solutions of eigenvalue problems*, MATH. COMPUT. 19 (1965), 462–466.

[25] K. K. HU, PH. G. KIRMSER, *A numerical solution of a nonlinear differential-integral equation for the optimal shape of the tallest column*, INT. J. ENGNG. SCI. 18 (1980), 333-339.

[26] A. D. IOFFE, V. M TIHOMIROV, *Theory of extremal problems*, NORTH-HOLAND, AMSTERDAM, 1979.

[27] J. B. KELLER, F. I. NIORDSON, *The Tallest Column. J. Math. Mech*. 16 (1966), 433–446.

[28] M. Y. KIM, A. I. MEHDI, M. M. ATTARD, *Finite strain theories of extensible and shear-flexible planar beams based on three different hypotheses of member forces*, INT. J. SOLIDS STRUCT. 193-194 (2020) 434–446.

[29] K. KOVARI, *Raumlishe Verzweingungsprobleme des dunnen elastischen Stabes mit endlichen Verformungen*, ING. ARCH. 37 (1969), 393–416.

[30] A. LIBAI, *Equations for the nonlinear planar deforrmation of beams*, J. APPL. MECH. 59 (1992), 1028–1030.

[31] O. L. MANGASARIAN, *Sufficient conditions for the optimal control of nonlinear systems*, SIAM J. CONTROL. 4 (1966), 139–152.

[32] C. M. MCCARTHY, *The tallest column: optimality revisited*, J. COMPUT. APPL. MATH. 101 (1999), 27–37.

[33] N. OLHOFF, B. ÅKESSON, *Minimum stiffness of optimally located supports for maximum value of column buckling loads*, STRUCT. OPTIMIZATION. 3 (1991), 163–175.

[34] J. D. RENTON, *Generalized theory applied to shear stiffness*, INT. J. SOLIDS STRUCT. 27 (1991), 1955–1967.

[35] S. SADIKU, *Buckling load optimization for heavy elastic columns: a perturbation approach*, STRUCT. MULTIDISC. OPTIM. 35 (2008), 447–452.

[36] A. P. SEYRANIAN, *A solution of a problem of Lagrange*, DOKL. AKAD. NAUK SSSR. 271 (1983), 337–340 (ENGLISH TRANSLATION: SOV. PHYS. DOKLADY. 28, 550–551.)

[37] A. P. SEYRANIAN, *The Lagrange problem on an optimal column*, UMAM. 2 (2003), 45–96.

[38] D. T. SPASIC, *Optimal design of elastic columns for maximum buckling load*, ACCEPTED FOR PUBLISHING IN VOLUME 3 OF STABILITY, 1997. VIBRATION AND CONTROL OF SYSTEMS, AVAILABLE FROM HTTP://MECHANICS.FTN.UNS.AC.RS/ZAPOSLENI/DRAGAN.SPASIC/PUBLICATIONS/DTSCHAPT.PDF.

[39] D. T. SPASIC, T. M. ATANACKOVIC, *Bimodal optimization of a compressed rotating rod*, ACTA MECH. 173 (2004), 77–87.

[40] D. T. SPASIC, V. B. GLAVARDANOV, *Does generalized elastica lead to bimodal optimal solutions*, INT. J. SOLIDS STRUCT. 46 (2009), 2939–2949.

[41] G. N. VANDERPLAATS, *Structural optimization*, PAST, PRESENT, AND FUTURE. AIAA J. 20, 992–1000.

[42] C. Y. WANG, B. DRACHMAN, *Stability of a heavy column with an end load*, J. APPL. MECH., 48 (1981), 668–669.