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Estimating the Gourava Sombor Index

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To the memory of Professor Ćemal Dolićanin, a man of virtue (1945–2023)

Abstract: The paper is concerned with the recently introduced vertex-degree-based graph invariant called Gourava Sombor index (*GSO*). Lower and upper bounds for *GSO* are obtained, in terms of Gourava index and the two Zagreb indices. For any graph, the upper bound is strict. Its improvements for special types of graphs are established.

Keywords: Gourava index, Sombor index, Gourava Sombor index, Zagreb index.

1 Introduction

In this paper we consider simple graphs. Let G be such a graph, possessing n vertices and m edges. Let $\mathbf{V}(G)$ and $\mathbf{E}(G)$ be the vertex and edge sets of G. Then, $|\mathbf{V}(G)| = n$ and $|\mathbf{E}(G)| = m$. By $uv \in \mathbf{E}(G)$ we denote the edge of G, connecting the vertices u and v. The degree (= number of first neighbors) of a vertex $u \in \mathbf{V}(G)$ is denoted by $d(u)$. Recall that if all vertices of a graph have mutually equal degrees, then this graph is said to be regular.

In order to avoid trivialities, throughout this paper it is assumed that G is connected. Note that for any vertex u of any connected graph, $d(u) \geq 1$.

For other graph-theoretical notions, the readers are referred to the textbooks [1, 11].

In the current mathematical and chemical literature, several dozens of vertex-degree-based graph invariants are studied. These are usually referred to as "VDB topological indices" [5, 13]. The general form of such a VDB topological index is

$$TI(G) = \sum_{uv \in \mathbf{E}(G)} F\big(d(u), d(v)\big)$$

where F is a function satisfying the condition $F(x,y) = F(y,x)$ for all $x, y \geq 1$.

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In this paper, we are concerned with the following VDB indices: the first [10] and the second Zagreb index [8]

$$Zg_1(G) = \sum_{uv\in\mathbf{E}(G)} \big[d(u)+d(v)\big] \qquad \text{and} \qquad Zg_2(G) = \sum_{uv\in\mathbf{E}(G)} d(u)\,d(v)$$

the Gourava index (more precisely: the first Gourava index) [12]

$$GO(G) = \sum_{uv\in\mathbf{E}(G)} \big[d(u)+d(v)+d(u)\,d(v)\big]$$

the Sombor index [6]

$$SO(G) = \sum_{uv\in\mathbf{E}(G)} \sqrt{d(u)^2+d(v)^2}$$

and the Gourava Sombor index [14]

$$GSO(G) = \sum_{uv\in\mathbf{E}(G)} \sqrt{\big[d(u)+d(v)\big]^2+\big[d(u)\,d(v)\big]^2}\,.$$

The Zagreb indices are the oldest and most thoroughly investigated VDB indices [2,3, 9,17]. Comparing the definitions of the Zagreb and Gourava indices, it is immediately seen that

$$GO(G) = Zg_1(G)+Zg_2(G)\,. \tag{1.1}$$

Thus the mathematical properties of GO can be directly deduced from those of Zg_1 and Zg_2.

Few years ago, the Sombor index was conceived using geometry-based considerations [6]. It soon attracted much attention and its numerous mathematical properties [15,18] and chemical applications [19,20] were established. The Gourava Sombor index is based on the idea to combine the algebraic forms of the Gourava and Sombor indices. Until now, only some chemical applications of GSO were reported [14], whereas its mathematical properties are examined here for the first time.

In what follows, we determine some basic mathematical properties of GSO, in particular its estimates in terms of GO, Zg_1, and Zg_2.

2 Elementary estimates

Theorem 2.1. *Let G be a connected graph on $n\geq 2$ vertices. Then*

$$\frac{1}{\sqrt{2}}\,GO(G) \leq GSO(G) < GO(G)$$

and

$$\frac{1}{\sqrt{2}}\,\big[Zg_1(G)+Zg_2(G)\big] \leq GSO(G) < Zg_1(G)+Zg_2(G)\,.$$

Equality on the left-hand side holds if and only if $n\geq 3$ and the graph G is regular of degree 2, i.e. G is the cycle C_n. The right-hand side inequality is strict.

Proof. We use the well known and often used [4, 7, 16] analytical inequality

$$\frac{1}{\sqrt{2}}(x+y) \le \sqrt{x^2+y^2} \le x+y \tag{2.1}$$

which holds for $x,y \ge 0$. Equality of the left-hand side holds if $x=y$. Equality on the right-hand side holds if $x=0$ or $y=0$ (or both).

Set $x = d(u)+d(v)$ and $y = d(u)\,d(v)$. Then by applying (2.1), we get

$$\begin{aligned} \frac{1}{\sqrt{2}}\left[d(u)+d(v)+d(u)\,d(v)\right] &\le \sqrt{\left[d(u)+d(v)\right]^2+\left[d(u)\,d(v)\right]^2} \\ &< d(u)+d(v)+d(u)\,d(v)\,. \end{aligned} \tag{2.2}$$

The left-hand side equality requires that $d(u)+d(v) = d(u)\,d(v)$. Vertex degrees are positive integers. Therefore, such equality holds if and only if $d(u)=d(v)=2$. The right-hand side inequality is strict since $d(u)+d(v)=0$ or $d(u)\,d(v)=0$ cannot occur.

Summation over all edges uv of the graph G, recalling the definition of the Gourava and Gourava Sombor indices, and taking into account the relation (1.1) results in Theorem 2.1. □

3 Improving the upper bound in Theorem 2.1

Define the auxiliary function

$$\Theta(x,y) = x+y+xy-\sqrt{(x+y)^2+(xy)^2}\,.$$

From the proof of Theorem 2.1 we know that $\Theta(x,y) > 0$ for $x,y \ge 1$.

Lemma 3.1. *For all $x,y \ge 1$,*

$$\frac{\partial\theta(x,y)}{\partial x} > 0 \qquad \text{and} \qquad \frac{\partial\theta(x,y)}{\partial y} > 0\,.$$

Proof. By direct calculation we get

$$\frac{\partial\theta(x,y)}{\partial x} = 1+y-\frac{x+y+xy^2}{\sqrt{(x+y)^2+(xy)^2}}\,. \tag{3.1}$$

Start now with the expression

$$2x^2y+2x^2y^2+4xy^2+2y^3+y^4$$

which evidently is positive-valued for $x,y \ge 1$. Therefore

$$\begin{aligned} &\left[2x^2y+2x^2y^2+4xy^2+2y^3+y^4\right]+\left[x^2+y^2+x^2y^4+2xy+2x^2y^2+2xy^3\right] \\ > \; & x^2+y^2+x^2y^4+2xy+2x^2y^2+2xy^3\,. \end{aligned}$$

Because

$$\begin{aligned} & \left[2x^2y+2x^2y^2+4xy^2+2y^3+y^4\right]+\left[x^2+y^2+x^2y^4+2xy+2x^2y^2+2xy^3\right] \\ = \;& (1+y)^2\left[(x+y)^2+(xy)^2\right] \end{aligned}$$

and

$$x^2+y^2+x^2y^4+2xy+2x^2y^2+2xy^3=(x+y+xy^2)^2$$

we get

$$(1+y)^2\left[(x+y)^2+(xy)^2\right]>(x+y+xy^2)^2$$

and

$$(1+y)\sqrt{(x+y)^2+(xy)^2}>x+y+xy^2$$

from which it straightforwardly follows that the right-hand side of Eq. (3.1) is positive-valued. This proves the first inequality in Lemma 3.1.

The second inequality in Lemma 3.1 follows by the symmetry of the function $\Theta(x,y)$. □

In what follows, we will say that an edge of the graph G, whose end vertices have degrees i and j, is an (i,j)-type edge.

Lemma 3.1 implies that in the case of the edge uv of the graph G, the right-hand side inequality (2.2) can be improved as

$$\sqrt{\left[d(u)+d(v)\right]^2+\left[d(u)\,d(v)\right]^2}\leq d(u)+d(v)+d(u)\,d(v)-\Theta(i,j) \tag{3.2}$$

where the parameters i and j should be chosen to be the smallest vertex degrees that the structure of G permits. Evidently, $i=j=1$ is the best choice, when $\Theta(1,1)=3-\sqrt{5}$

By summation of (3.2) over all edges of the graph G, we arrive at:

Theorem 3.1. *Let G be a connected graph with $n\geq 2$ vertices and m edges. Then*

$$GSO(G)\leq GO(G)-(3-\sqrt{5})m$$

i.e.,

$$GSO(G)\leq Zg_1(G)+Zg_2(G)-(3-\sqrt{5})m\,.$$

Equality holds if and only if all edges of the graph are of $(1,1)$-type, which for connected graphs can happen only if $n=2$, $m=1$, i.e., if the graph G is the 2-vertex path.

By a analogous reasoning we obtain the following improvements of Theorem 2.1:

Theorem 3.2. *Let G be a connected graph with $n\geq 3$ vertices and m edges. Then*

$$GSO(G)\leq GO(G)-(5-\sqrt{13})m$$

i.e.,

$$GSO(G)\leq Zg_1(G)+Zg_2(G)-(5-\sqrt{13})m\,.$$

Equality holds if and only if all edges of the graph are of of $(1,2)$-type, which for connected graphs can happen only if $n=3$, $m=2$, i.e., if the graph G is the 3-vertex path.

Theorem 3.3. *Let G be a connected graph with $n \geq 3$ vertices and m edges, and without vertices of degree 1. Then*

$$GSO(G) \leq GO(G) - (8 - \sqrt{32})m$$

i.e.,

$$GSO(G) \leq Zg_1(G) + Zg_2(G) - (8 - \sqrt{32})m\,.$$

Equality holds if and only if all edges of the graph are of $(2,2)$-type, which means that G is the n-vertex cycle C_n, $n \geq 3$.

Theorems 3.1 and 3.3 can be generalized as:

Theorem 3.4. *Let G be a connected graph with n vertices and m edges, and let its smallest vertex degree be $\delta \geq 1$. Then*

$$GSO(G) \leq GO(G) - (\delta + 2 - \sqrt{\delta^2 + 4})\,\delta m$$

i.e.,

$$GSO(G) \leq Zg_1(G) + Zg_2(G) - (\delta + 2 - \sqrt{\delta^2 + 4})\,\delta m\,.$$

Equality holds if and only if all edges of the graph are of (δ, δ)-type, which means that G is a regular graph of degree δ.

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