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On Some Selection Principles in the Space of *G*-permutation Degree

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Dedicated to the memory of Professor Ćemal Dolićanin (1945–2023)

Abstract: In this paper, we study several selection and star selection principles in the space of G-permutation degree $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$. In particular, we consider such selection properties $\mathscr{P}$ for which $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ has $\mathscr{P}$ if and only if X has $\mathscr{P}$.

Keywords: space of G-permutation degree, (star) selection principles, ω-Rothberger space, Alster space, set star selection principles.

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1 Introduction

Recently, the selection and star selection principles in topology have been studied by many authors ([5, 6, 14, 15, 24]). They considered several selection (star selection) properties and studied the relations between a space X satisfying such a property and its hyperspaces with the Vietoris and other topologies.

In [8, 9] V. V. Fedorchuk and V. V. Filippov investigated the functor of G-permutation degree and it was proved that the functor of G-permutation degree $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}$ is a normal functor in the category of compact spaces and their continuous mappings.

In recent years researches were interested in the theory of cardinal invariants, homotopy properties, some classes of topological spaces and their behavior under the influence of various covariant functors, in particular under influence of the functor of G-permutation degree (see [4, 16, 17, 18, 19, 20, 21]).

In [4, 16], it was studied index of boundedness, uniform connectedness and homotopy properties of the space of G-permutation degree. In [16], it was shown that the functor

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$\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}$ preserves the homotopy and the retraction of topological spaces. In addition, it was proved that if the spaces X and Y are homotopically equivalent, then the spaces $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ and $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}Y$ are also homotopically equivalent., i.e. that the functor $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}$ is a covariant homotopy functor.

In [17, 18, 19], some tightness-type properties, network-type properties and Lindelǒf-type properties of the space of G-permutation degree have been studied. For example, in [19] it was proved that the functor $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}$ preserves *cs*-network, *cs**-network, *cn*-network and *ck*-network of topological spaces. The papers [20, 21] are devoted to the investigation of some classes of topological spaces (such as developable spaces, Moore spaces, M_1-spaces, M_2-spaces, Lašnev spaces and Nagata spaces) in the space of G-permutation degree.

In this paper, we study the relation between a space X (and its power X^nP) satisfying certain selection (or star selection) properties and the space of G-permutation degree $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$. We prove:

(1) If X^n is an almost Menger space, then so is $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$;

(2) If X^n is a star-Menger space, then so is $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$;

(3) X is an ω-Rothberger space (an Alster space) if and only if $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ is ω-Rothberger (Alster);

(4) If X^n is set star-Menger (set strongly star Menger) space, then so is $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$;

(6) A space X^n is weakly strongly star-Menger if and only if the space $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ is weakly strongly star-Menger.

Throughout this paper all spaces are assumed to be T_2.

2 Preliminary notes

The set of all non-empty closed subsets of a topological space X is denoted by $\exp X$. The family of all sets of the form

$$O\langle U_1, U_2, \dots, U_n\rangle = \big\{F:\ F \in \exp X, F \subset \bigcup_{i=1}^{n} U_i, F \cap U_i \neq \emptyset, i = 1, \dots, n\big\},$$

where $U_1, U_2, \dots, U_n$ are open subsets of X, generates a base of the topology on the set $\exp X$. This topology is called the *Vietoris topology*. The set $\exp X$ with the Vietoris topology is called the *exponential space* or the *hyperspace* of the space X. We put

$$\mathsf{exp}_{\mathsf{n}}X = \{F \in \exp X :\ |F| \leq n\}\ [9].$$

Let S_n be the group of all permutations of the set $X = \{1,2,\dots,n\}$ and let G be a subgroup of S_n. Let X^n be the n-th power of a topological space X. The permutation group G acts on X^n as permutation of coordinates: two points $(x_1,x_2,\dots,x_n),(y_1,y_2,\dots,y_n) \in X^n$ are considered to be *G-equivalent* if there exists a permutation $\sigma \in G$ such that $y_i = x_{\sigma(i)}$. The set of all orbits of this action with the quotient topology is denoted by $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$. The quotient mapping, denoted by $\pi^s_{n,G} : X^n \to \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$, is defined by

$$\pi^s_{n,G}((x_1,x_2,\dots,x_n)) = [(x_1,x_2,\dots,x_n)]_G,$$

for every $(x_1,x_2,\dots,x_n) \in X^n$

Thus, the points of the space $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ are finite subsets (equivalence classes) of the product X^n. The space $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ is called the *space of n-G-permutation degree* (or simply the *space of G-permutation degree*) of the space X When $G = S_n$ we omit G in the previous notations.

Let $f : X \to Y$ be a continuous mapping. One defines the mapping $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}f : \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X \to \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}Y$: for an equivalence class $[(x_1,x_2,\dots,x_n)]_G \in \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ we put

$$\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}f[(x_1,x_2,\dots,x_n)]_G = [(f(x_1),f(x_2),\dots,f(x_n))]_G.$$

In this way one obtains a normal functor $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}$ in the category of compact spaces and their continuous mappings. This functor is called the *functor of G-permutation degree*.

Equivalence relations by which we obtained spaces $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ and $\exp_{\mathsf{n}}X$ are called the *symmetric* and *hypersymmetric* equivalence relations, respectively. Symmetrically equivalent points in X^n are hypersymmetrically equivalent, but the inverse is not true, in general.

The G-symmetric equivalence class $[(x_1,x_2,\dots,x_n)]_G$ uniquely determines the hypersymmetric equivalence class $[(x_1,x_2,\dots,x_n)]^{hc}$ containing it. The mapping

$$\pi^h_{n,G} : \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X \to \exp_{\mathsf{n}}X,$$

represents the functor $\exp_{\mathsf{n}}$ as the factor functor of the functor $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}$ [8, 9]. Note that the spaces $\exp_2 X$ and $\mathsf{SP}^{2}_{\mathsf{G}}X$ are homeomorphic; there are examples showing that for $n \geq 3$ the spaces $\exp_{\mathsf{n}}X$ and $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ need not be homeomorphic [8, 9].

3 Results

3.1 Selection principles

This subsection is devoted to certain selective properties of the space of G-permutation degree.

In general Menger-type properties are not preserved under the functor $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}$, in particular the nth power of a Menger-type space need not be of the same type.

A space X is *almost Menger* [13, 22] if for every sequence $\{\mathscr{U}_m\}_{m\in\mathbb{N}}$ of open covers of X there is a sequence $\{\mathscr{V}_m\}_{m\in\mathbb{N}}$ such that $\mathscr{V}_m$ is a finite subset of $\mathscr{U}_m$ for every $m \in \mathbb{N}$ and $\bigcup_{m\in\mathbb{N}} \bigcup\{\overline{V} : V \in \mathscr{V}_m\} = X$.

Theorem 3.1 *If X^n is an almost Menger space, then so is $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$.*

Proof. Let X^n be an almost Menger space and $\{\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}\mathscr{U}_m\}_{m\in\mathbb{N}}$ be a sequence of open covers of $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$. Put $\mathscr{U}_m = \{(\pi^s_{n,G})^{\leftarrow}(\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}U) : \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}U \in \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}\mathscr{U}_m\}$ for every $m \in \mathbb{N}$. Since $\pi^s_{n,G}(X^n) = \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ and the mapping $\pi^s_{n,G}$ is continuous, $\{\mathscr{U}_m\}_{m\in\mathbb{N}}$ is a sequence of open covers of X^n. Since X^n is an almost Menger space, there is a sequence $\{\mathscr{V}_m\}_{m\in\mathbb{N}}$ of finite sets such that $\mathscr{V}_m$ is a subset of $\mathscr{U}_m$ for every $m \in \mathbb{N}$ and $\bigcup_{m\in\mathbb{N}} \bigcup\{\overline{V} : V \in \mathscr{V}_m\} = X^n$. Put

$\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}\mathscr{V}_m = \{\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}V : \pi^s_{n,G}(V) = \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}V, V \in \mathscr{V}_m\}$ for every $m \in \mathbb{N}$. We have that $\{\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}\mathscr{V}_m\}_{m\in\mathbb{N}}$ is a sequence of finite subsets of $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}\mathscr{U}_m$ for every $m \in \mathbb{N}$. On the other hand, we have

$$\begin{aligned} \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X &= \pi^s_{n,G}(X^n) = \pi^s_{n,G}\left(\bigcup_{m\in\mathbb{N}}\bigcup\{\overline{V} : V \in \mathscr{V}_m\}\right) \\ &= \bigcup_{m\in\mathbb{N}}\bigcup\{\pi^s_{n,G}(\overline{V}) : V \in \mathscr{V}_m\} \\ &\subset \bigcup_{m\in\mathbb{N}}\bigcup\left\{\overline{\pi^s_{n,G}(V)} : V \in \mathscr{V}_m\right\}. \end{aligned}$$

Finally, we have that $\bigcup_{m\in\mathbb{N}}\bigcup\{\overline{\pi^s_{n,G}(V)} : V \in \mathscr{V}_m\} = \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$, and it means that the space $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ is almost Menger. Theorem 3.1 is proved. □

A space X is *weakly set-Menger* (resp., *almost set-Menger*), if for every nonempty subset A of X and every sequence $\{\mathscr{U}_m\}_{m\in\mathbb{N}}$ of open sets in X such that for every $m \in \mathbb{N}$, $\overline{A} \subset \cup\mathscr{U}_m$, there exists a sequence $\{\mathscr{V}_m\}_{m\in\mathbb{N}}$ such that for every $m \in \mathbb{N}$, $\mathscr{V}_m$ is a finite subset of $\mathscr{U}_m$ and $A \subset \overline{\bigcup_{m\in\mathbb{N}}\bigcup\mathscr{V}_m}$ (resp., $A \subset \bigcup_{m\in\mathbb{N}}\overline{\bigcup\mathscr{V}_m}$).

Similarly to the proof of Theorem 3.1 we can prove the following result.

Theorem 3.2 *If X^n is a weakly set-Menger space (resp., an almost set-Menger space), then so is $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$.*

Recall that an open cover $\mathscr{U}$ of a space X is said to be an *ω-cover* if every finite subset of X is contained in a member of $\mathscr{U}$. A space X is said to be *ω-Rothberger* if for every sequence $\{\mathscr{U}_m\}_{m\in\mathbb{N}}$ of ω-covers of X there is a sequence $\{U_m\}_{m\in\mathbb{N}}$ such that for every $m \in \mathbb{N}$, $U_m \in \mathscr{U}_m$ and $\{U_m : m \in \mathbb{N}\}$ is an ω-cover of X.

Theorem 3.3 *For a space X, the space $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ is ω-Rothbrger if and only if X is ω-Rothberger.*

Proof. ($\Rightarrow$) Let $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ be an ω-Rothberger space. We first prove that the space X^n is ω-Rothberger. Let $\{\mathscr{U}_m\}_{m\in\mathbb{N}}$ be a sequence of ω-covers of X^n. We prove that $\pi^s_{n,G}(\mathscr{U}_m)$ is an ω-cover of $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ for every $m \in \mathbb{N}$. First, the mapping $\pi^s_{n,G} : X^n \to \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ is open so that $\pi^s_{n,G}(\mathscr{U}_m)$ is an open cover of $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$. Let F be a finite subset of $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$. As $\pi^s_{n,G}$ is a finite-to-one mapping, the set $(\pi^s_{n,G})^{\leftarrow}(F)$ is a finite subset of X^n. Therefore, there exists $U \in \mathscr{U}_m$ such that $(\pi^s_{n,G})^{\leftarrow}(F)$ is contained in U. It follows that $F \subset \pi^s_{n,G}(U) \in \pi^s_{n,G}(\mathscr{U}_m)$. In other words, $\{\pi^s_{n,G}(\mathscr{U}_m)\}_{m\in\mathbb{N}}$ is a sequence of ω-covers of $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$. Thus one can choose a $\pi^s_{n,G}(U_m) \in \pi^s_{n,G}(\mathscr{U}_m)$, $m \in \mathbb{N}$, such that $\{\pi^s_{n,G}(U_m) : m \in \mathbb{N}\}$ is an ω-cover of $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$. This means that the sequence $\{U_m\}_{m\in\mathbb{N}}$ witnesses for $\{\mathscr{U}_m\}_{m\in\mathbb{N}}$ that X^n is an ω-Rothberger space. It follows from this that X is ω-Rothberger as the image of X^n under the projection mapping.

($\Leftarrow$) Let X be an ω-Rorhberger space. It was shown in [10] that all finite powers of an ω-Rothberger space are also ω-Rothberger, hence X^n is ω-Rothbrger. We prove that $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ is an ω-Rothberger space. Let $\{\mathscr{U}_m\}_{m\in\mathbb{N}}$ be a sequence of ω-covers of $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$. Since the mapping $\pi^s_{n,G}$ is finite-to-one, one can easily conclude that $\{(\pi^s_{n,G})^{\leftarrow}(\mathscr{U}_m)\}_{m\in\mathbb{N}}$ is a sequence of ω-covers of X^n. Since X^n is ω-Rothberger, there is a sequence $\{(\pi^s_{n,G})^{\leftarrow}(U_m)\}_{m\in\mathbb{N}}$ such that for every $m\in\mathbb{N}$, $(\pi^s_{n,G})^{\leftarrow}(U_m)\in(\pi^s_{n,G})^{\leftarrow}(\mathscr{U}_m)$ and $\{(\pi^s_{n,G})^{\leftarrow}(U_m) : m\in\mathbb{N}\}$ is an ω-cover of X^n. Then the sequence $\{U_m\}_{m\in\mathbb{N}}$ shows that $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ is an ω-Rothbrger space. □

A space X is said to be *ω-Menger* if for every sequence $\{\mathscr{U}_m\}_{m\in\mathbb{N}}$ of ω-covers of X there is a sequence $\{\mathscr{V}_m\}_{m\in\mathbb{N}}$ such that for every $m\in\mathbb{N}$, $\mathscr{V}_m$ is finite subset of $\mathscr{U}_m$ and $\bigcup_{m\in\mathbb{N}}\mathscr{V}_n$ is an ω-cover of X.

Similarly to the proof of the previous theorem, by using the fact that all finite powers of an ω-Menger space are ω-Menger [10], one can prove the following.

Theorem 3.4 *For a space X, the space $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ is ω-Menger if and only if X is ω-Menger.*

The following notation we borrow from [1]. The symbol $\mathscr{G}$ denotes the collection of all covers of a space X by G_δ subsets of X. The symbol $\mathscr{G}_K$ denotes the collection of all $\mathscr{U}\in\mathscr{G}$ such that $X\notin\mathscr{U}$ and for every compact subset C of X there is $U\in\mathscr{U}$ containing C; covers from $\mathscr{G}_K$ are called *Alster covers*.

A space X is said to be an *Alster space* [1, 3] if for every sequence $\{\mathscr{U}_m\}_{m\in\mathbb{N}}$ of elements of $\mathscr{G}_K$ there is a sequence $\{U_m\}_{m\in\mathbb{N}}$ such that $U_m\in\mathscr{U}_m$ for every $m\in\mathbb{N}$ and $\{U_m : m\in\mathbb{N}\}\in\mathscr{G}$.

Theorem 3.5 *A space X is an Alster space if and only if $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ is so.*

Proof. ($\Rightarrow$) Let X be an Alster space. By [2, Theorem 4.5] (see also [1, 3]) the space X^n is also an Alster space. We are going now to prove that $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ is an Alster space. Let $\{\mathscr{U}_m\}_{m\in\mathbb{N}}$ be a sequence of Alster covers of $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$. Since the mapping $\pi^s_{n,G}$ is closed and finite-to-one, hence perfect, it is easy to check that $\{(\pi^s_{n,G})^{\leftarrow}(\mathscr{U}_m)\}_{m\in\mathbb{N}}$ is a sequence of Alster covers of X^n. Since X^n is an Alster space there is a sequence $\{(\pi^s_{n,G})^{\leftarrow}(U_m)\}_{m\in\mathbb{N}}\in\mathscr{G}$ such that for every $m\in\mathbb{N}$, $(\pi^s_{n,G})^{\leftarrow}(U_m)\in(\pi^s_{n,G})^{\leftarrow}(\mathscr{U}_m)$. Then the sequence $\{U_m\}_{m\in\mathbb{N}}$ witnesses for $\{\mathscr{U}_m\}_{m\in\mathbb{N}}$ that $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ is an Alster space.

($\Leftarrow$) Let now $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ be an Alster space and let $\{\mathscr{V}_m\}_{m\in\mathbb{N}}$ be a sequence of Alster covers of X^n. We prove that $\pi^s_{n,G}(\mathscr{V}_m)$ is an Alster cover of $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ for every $m\in\mathbb{N}$. Let C be a compact subset of $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$. As $\pi^s_{n,G}$ is a perfect mapping, the set $(\pi^s_{n,G})^{\leftarrow}(C)$ is a compact subset of X^n. Therefore, there is some $V\in\mathscr{V}_m$ such that $(\pi^s_{n,G})^{\leftarrow}(C)\subset V$ and thus $C\subset\pi^s_{n,G}(V)\in\pi^s_{n,G}(\mathscr{V}_m)$. So, $\{\pi^s_{n,G}(\mathscr{V}_m)\}_{m\in\mathbb{N}}$ is a sequence of Alster covers of $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$. We can find for every $m\in\mathbb{N}$ an element $\pi^s_{n,G}(V_m)\in\pi^s_{n,G}(\mathscr{V}_m)$ such that $\{\pi^s_{n,G}(V_m) : m\in\mathbb{N}\}\in\mathscr{G}$. The sequence $\{V_m\}_{m\in\mathbb{N}}$ guarantees for $\{\mathscr{V}_m\}_{m\in\mathbb{N}}$ that X^n is an Alster space.

Finally, we use the projection $pr : X^n\to X$ to conclude that X is an Alster space. □

3.2 Star selection principles

In this subsection we consider some star selection principles in the space of G-permutation degree. If A is a subset of a space X and $\mathscr{A}$ is collection of subset of X, then $\mathrm{St}(A,\mathscr{A}) = \cup\{B \in \mathscr{A} : B \cap A \neq \emptyset\}$.

A space X is called *star-Menger* [11], if for every sequence $\{\mathscr{U}_m\}_{m\in\mathbb{N}}$ of open covers of X there is a sequence $\{\mathscr{V}_m\}_{m\in\mathbb{N}}$ such that $\mathscr{V}_m$ is a finite subset of $\mathscr{U}_m$ for every $m \in \mathbb{N}$, and $\{\mathrm{St}(\cup\mathscr{V}_m, \mathscr{U}_m)\}_{m\in\mathbb{N}}$ is a cover of X.

Theorem 3.6 *If X^n is a star-Menger space, then so is $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$.*

Proof. Assume that X^n is a star-Menger space. Let $\{\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}\mathscr{U}_m\}_{m\in\mathbb{N}}$ be a sequence of open covers of $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$. Put $\mathscr{U}_m = \{(\pi^s_{n,G})^{\leftarrow}(\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}U) : \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}U \in \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}\mathscr{U}_m\}$ for every $m \in \mathbb{N}$. Since $\pi^s_{n,G}(X^n) = \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ and the mapping $\pi^s_{n,G}$ is continuous, $\{\mathscr{U}_m\}_{m\in\mathbb{N}}$ is a sequence of open covers of X^n. Thus there is a sequence $\{\mathscr{V}_m\}_{m\in\mathbb{N}}$ of finite subsets of $\mathscr{U}_m$, $m \in \mathbb{N}$, such that $\bigcup_{m\in\mathbb{N}}\mathrm{St}(\bigcup\mathscr{V}_m, \mathscr{U}_m) = X^n$. Let $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}\mathscr{V}_m = \pi^s_{n,G}(\mathscr{V}_m)$. Clearly, $\{\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}\mathscr{V}_m\}_{m\in\mathbb{N}}$ is a sequence of finite subsets of $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}\mathscr{U}_m$, $m \in \mathbb{N}$. It is easily checked that $\left\{\mathrm{St}(\bigcup\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}\mathscr{V}_m, \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}\mathscr{U}_m) : m \in \mathbb{N}\right\}$ is a cover of $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$. It shows $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ is a star-Menger space. □

A space X is called *almost star-Menger* [22], if for every sequence $\{\mathscr{U}_m\}_{m\in\mathbb{N}}$ of open covers of X there is a sequence $\{\mathscr{V}_m\}_{m\in\mathbb{N}}$ such that $\mathscr{V}_m$ is a finite subset of $\mathscr{U}_m$ for every $m \in \mathbb{N}$, and $\{\overline{\mathrm{St}(\cup\mathscr{V}_m, \mathscr{U}_m)} : m \in \mathbb{N}\}$ is a cover of X.

Theorem 3.7 *If X^n is an almost star-Menger space, then so is $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$.*

Proof. Let X^n be an almost star-Menger space and $\{\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}\mathscr{U}_m\}_{m\in\mathbb{N}}$ be a sequence of open covers of $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$. Put $\mathscr{U}_m = \{(\pi^s_{n,G})^{\leftarrow}(\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}U) : \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}U \in \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}\mathscr{U}_m\}$ for every $m \in \mathbb{N}$. Since $\pi^s_{n,G}(X^n) = \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ and the mapping $\pi^s_{n,G}$ is continuous, $(\mathscr{U}_m)_{m\in\mathbb{N}}$ is a sequence of open covers of X^n. Since X^n is an almost star-Menger space, there is a sequence $\{\mathscr{V}_m\}_{m\in\mathbb{N}}$ of finite sets such that $\mathscr{V}_m$ is a subset of $\mathscr{U}_m$ for every $m \in \mathbb{N}$ and $\bigcup_{m\in\mathbb{N}}\overline{\mathrm{St}(\cup\mathscr{V}_m, \mathscr{U}_m)} = X^n$. Let $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}\mathscr{V}_m = \{\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}U : (\pi^s_{n,G})^{\leftarrow}(\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}U) \in \mathscr{V}_m\}$. If $\mathbf{x} = (x_1, \ldots, x_n) \in X^n$, then it follows from $(\pi^s_{n,G})^{\leftarrow}(\bigcup\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}\mathscr{V}_m) = \bigcup\mathscr{V}_m$ that there is $k \in \mathbb{N}$ such that $\mathbf{x} \in \overline{\mathrm{St}((\pi^s_{n,G})^{\leftarrow}(\bigcup\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}\mathscr{V}_k), \mathscr{U}_k)}$. If $[\mathbf{x}]_G = \pi^s_{n,G}(\mathbf{x}) \in \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$, then

$$
\begin{aligned}
[\mathbf{x}]_G &\in \pi^s_{n,G}\left(\overline{\mathrm{St}\left((\pi^s_{n,G})^{\leftarrow}(\bigcup\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}\mathscr{V}_k), \mathscr{U}_k\right)}\right) \\
&\subseteq \overline{\pi^s_{n,G}\left(\mathrm{St}\left((\pi^s_{n,G})^{\leftarrow}(\bigcup\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}\mathscr{V}_k), \mathscr{U}_m\right)\right)} \\
&\subseteq \overline{\mathrm{St}\left(\bigcup\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}\mathscr{V}_k, \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}\mathscr{U}_k\right)}.
\end{aligned}
$$

On the other hand, suppose that $(\pi^s_{n,G})^{\leftarrow}(\bigcup\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}\mathscr{V}_m) \cap (\pi^s_{n,G})^{\leftarrow}(\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}U) \neq \emptyset$. Then also $\pi^s_{n,G}((\pi^s_{n,G})^{\leftarrow}(\bigcup\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}\mathscr{V}_m)) \cap \pi^s_{n,G}((\pi^s_{n,G})^{\leftarrow}(\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}U)) \neq \emptyset$, so $\bigcup\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}\mathscr{V}_m \cap \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}U \neq \emptyset$. It shows that the space $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ is almost star-Menger. Theorem 3.7 is proved. □

A space X is called *strongly star-Menger* [11, 12] if for every sequence $\{\mathscr{U}_m\}_{m\in\mathbb{N}}$ of open covers of X, there exists a sequence $\{F_m\}_{m\in\mathbb{N}}$ of finite subsets of X such that $\bigcup_{m\in\mathbb{N}} \mathrm{St}(F_m, \mathscr{U}_m) = X$.

By a small modification of the proof of Theorem 3.6 we can prove the following.

Theorem 3.8 *If X^n is a strongly star-Menger space, then so is $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$.*

A space X is called *set star-Menger* if for every nonempty subset A of X and every sequence $\{\mathscr{U}_m\}_{m\in\mathbb{N}}$ of collections of open sets in X such that $\overline{A} \subset \cup\mathscr{U}_m$, $m \in \mathbb{N}$, there exists a sequence $\{\mathscr{V}_m\}_{m\in\mathbb{N}}$ such that $\mathscr{V}_m$ is a finite subset of $\mathscr{U}_m$ for every $m \in \mathbb{N}$ and $A \subset \bigcup_{m\in\mathbb{N}} \mathrm{St}(\cup\mathscr{V}_m, \mathscr{U}_m)$;

Theorem 3.9 *If X^n is a set star-Menger space, then so is $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$.*

Proof. Assume that X^n is a set star-Menger space. Let $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}A$ be any nonempty subset of $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ and $\{\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}\mathscr{V}_m\}_{m\in\mathbb{N}}$ be a sequence of open covers of $\overline{\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}A}$. Put $A = (\pi^s_{n,G})^{\leftarrow}(\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}A)$. By the continuity of the mapping $\pi^s_{n,G}$ for every $m \in \mathbb{N}$, $\mathscr{V}_m = \{(\pi^s_{n,G})^{\leftarrow}(\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}V) : \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}V \in \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}\mathscr{V}_m\}$ is a collection of open sets in X^n with

$$\overline{A} = \overline{(\pi^s_{n,G})^{\leftarrow}(\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}A)} \subset (\pi^s_{n,G})^{\leftarrow}\overline{(\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}A)} \subset (\pi^s_{n,G})^{\leftarrow}(\cup\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}\mathscr{V}_m) = \cup\mathscr{V}_m.$$

Since X^n is a set star-Menger space, there exists a sequence $\{\mathscr{U}_m\}_{m\in\mathbb{N}}$ such that $\mathscr{U}_m$ is a finite subset of $\mathscr{V}_m$ for every $m \in \mathbb{N}$, and $A \subset \bigcup_{m\in\mathbb{N}} \mathrm{St}(\cup\mathscr{U}_m, \mathscr{V}_m)$. Put $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}\mathscr{U}_m = \{\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}V : (\pi^s_{n,G})^{\leftarrow}(\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}V) \in \mathscr{U}_m\}$. Consequently, for every $m \in \mathbb{N}$, $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}\mathscr{U}_m$ is a finite subset of $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}\mathscr{V}_m$ and

$$\begin{aligned}
\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}A &\subset \pi^s_{n,G}\Big(\bigcup_{m\in\mathbb{N}} \mathrm{St}(\cup\mathscr{U}_m, \mathscr{V}_m)\Big) \\
&\subset \bigcup_{m\in\mathbb{N}} \mathrm{St}(\cup\pi_{n,G}(\{(\pi^s_{n,G})^{\leftarrow}(\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}V) : \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}V \in \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}\mathscr{U}_m\}), \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}\mathscr{V}_m) \\
&= \bigcup_{m\in\mathbb{N}} \mathrm{St}(\cup\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}\mathscr{U}_m, \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}\mathscr{V}_m).
\end{aligned}$$

It means that $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ is a set star-Menger space. Theorem 3.9 is proved. □

A space X is called *set strongly star-Menger* if for every nonempty subset A of X and every sequence $\{\mathscr{U}_m\}_{m\in\mathbb{N}}$ of collections of open sets in X such that $\overline{A} \subset \cup\mathscr{U}_m$, there exists a sequence $\{F_m\}_{m\in\mathbb{N}}$ of finite subsets of $\overline{A}$ such that $A \subset \bigcup_{m\in\mathbb{N}} \mathrm{St}(F_m, \mathscr{U}_m)$.

We can also prove the following theorem by a similar way to the proof of Theorem 3.9.

Theorem 3.10 *If X^n is a set strongly star-Menger space, then so is $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$.*

The following lemma is well known [7].

Lemma 3.1 *A continuous mapping $f : X \to Y$ is closed if and only if for every point $y \in Y$ and every open set $U \subset X$ which contains $f^{\leftarrow}(y)$, there exists a neighbourhood V of the point y such that $f^{\leftarrow}(V) \subset U$.*

We say that a space X is *nearly set strongly star-Menger* [15] if for every $A \subset X$ and every sequence $\{\mathscr{U}_m\}_{m\in\mathbb{N}}$ of open covers of X there exists a sequence $\{F_m\}_{m\in\mathbb{N}}$ of finite subsets of X such that $A \subset \bigcup_{m\in\mathbb{N}} \mathrm{St}(F_m, \mathscr{U}_m)$.

Theorem 3.11 *If the space $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ is set strongly star-Menger, then the space X^n is nearly set strongly star-Menger.*

Proof. Suppose that A is a nonempty subset of X^n and $\{\mathscr{U}_m\}_{m\in\mathbb{N}}$ is a sequence of open covers of X^n. Consider the set $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}A = \pi^s_{n,G}(A) \subset \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$. Let $[\mathbf{x}]_G = [(x_1,x_2,...,x_n)]_G \in \overline{\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}A}$. The set $(\pi^s_{n,G})^{\leftarrow}([\mathbf{x}]_G)$ is a finite subset of X^n, and for every $m \in \mathbb{N}$ there is a finite subset $\mathscr{U}_{m,[\mathbf{x}]_G}$ of $\mathscr{U}_m$ such that $(\pi^s_{n,G})^{\leftarrow}([\mathbf{x}]_G) \subset \mathscr{U}_{m,[\mathbf{x}]_G}$ and $U \cap (\pi^s_{n,G})^{\leftarrow}([\mathbf{x}]_G) \neq \emptyset$ for every $U \in \mathscr{U}_{m,[\mathbf{x}]_G}$. Since the mapping $\pi^s_{n,G}$ is closed, by Lemma 3.1, there is an open neighborhood $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}V_{m,[\mathbf{x}]_G}$ of $[\mathbf{x}]_G$ in $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ such that $(\pi^s_{n,G})^{\leftarrow}(\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}V_{m,[\mathbf{x}]_G}) \subset \cup\{U : U \in \mathscr{U}_{m,[\mathbf{x}]_G}\}$. Since the mapping $\pi^s_{n,G}$ is open, we have that $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}V_{m,[\mathbf{x}]_G} \subseteq \cap\{\pi^s_{n,G}(U) : U \in \mathscr{U}_{m,[\mathbf{x}]_G}\}$. Put $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}\mathscr{V}_m = \{\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}V_{m,[\mathbf{x}]_G} : [\mathbf{x}]_G \in \overline{\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}A}\}$ for any $m \in \mathbb{N}$. It is known that $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}\mathscr{V}_m$ is an open cover of $\overline{\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}A}$. Since $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ is set strongly star-Menger, there exists a sequence $\{\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}F_m\}_{m\in\mathbb{N}}$ of finite subsets of $\overline{\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}A}$ such that $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}A \subset \bigcup_{m\in\mathbb{N}} \mathrm{St}(\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}F_m, \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}\mathscr{V}_m)$. Clearly, the sequence $F_m = \{(\pi^s_{n,G})^{\leftarrow}(\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}F_m)\}_{m\in\mathbb{N}}$ is a sequence of finite subsets of X^n.

We now prove that $A \subset \bigcup_{m\in\mathbb{N}} \mathrm{St}(F_m, \mathscr{U}_m)$. Let $\mathbf{x} = (x_1,x_2,...,x_n) \in A$. Then there exist $m \in \mathbb{N}$ and $[\mathbf{y}]_G \in \overline{\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}A}$ such that $\pi^s_{n,G}(\mathbf{x}) \in \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}V_{m,[\mathbf{y}]_G}$ and $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}V_{m,[\mathbf{y}]_G} \cap F_m \neq \emptyset$. Since $\mathbf{x} \in (\pi^s_{n,G})^{\leftarrow}(\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}V_{m,[\mathbf{y}]_G}) \subset \bigcup\{U : \mathscr{U}_{m,[\mathbf{x}]_G}\}$, there is $U \in \mathscr{U}_{m,[\mathbf{x}]_G}$ such that $\mathbf{x} \in U$. Then $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}V_{m,[\mathbf{y}]_G} \subset \pi^s_{n,G}(U)$, which means $U \cap F_m \neq \emptyset$. Hence, $\mathbf{x} \in \mathrm{St}(F_m, \mathscr{U}_m)$ and it follows that $A \subset \bigcup_{m\in\mathbb{N}} \mathrm{St}(F_m, \mathscr{U}_m)$. Theorem 3.11 is proved. □

A space X is called *weakly strongly star-Menger* [23] if for every sequence $\{\mathscr{U}_m\}_{m\in\mathbb{N}}$ of open covers of X, there is a sequence $\{F_m\}_{m\in\mathbb{N}}$ of finite subsets of X such that $\overline{\bigcup_{m\in\mathbb{N}} \mathrm{St}(F_m, \mathscr{U}_m)} = X$. The weakly strongly star-Mengerness is weaker than the strongly star-Mengerness.

Theorem 3.12 *A space X^n is weakly strongly star-Menger if and only if the space $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ is weakly strongly star-Menger.*

Proof. $(\Rightarrow)$ Suppose that X^n is a weakly strongly star-Menger space. Let $\{\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}\mathscr{U}_m\}_{m\in\mathbb{N}}$ be a sequence of open covers of $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$. Put

$$\mathscr{U}_m = \{(\pi^s_{n,G})^{\leftarrow}(\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}U) : \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}U \in \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}\mathscr{U}_m\}, \qquad m \in \mathbb{N}.$$

Clearly, $\{\mathscr{U}_m\}_{m\in\mathbb{N}}$ is a sequence of open covers of X^n. Since X^n is a weakly strongly star-Menger space, there is a sequence $\{F_m\}_{m\in\mathbb{N}}$ of finite subsets of X^n such that $\overline{\bigcup_{m\in\mathbb{N}} \mathrm{St}(F_m, \mathscr{U}_m)} = X^n$. Let $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}F_m = \pi^s_{n,G}(F_m)$ for every $m \in \mathbb{N}$. Then $\{\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}F_m\}_{m\in\mathbb{N}}$ is a sequence of finite subsets of $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$. If $[\mathbf{x}]_G = \pi^s_{n,G}(\mathbf{x}) \in \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$, then

$$\begin{aligned} [\mathbf{x}]_G \quad &\in \quad \pi^s_{n,G}\overline{(\mathrm{St}((\pi^s_{n,G})^{\leftarrow}(\bigcup \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}F_m), \mathscr{U}_m))} \\ &\subseteq U \quad \overline{\pi^s_{n,G}(\mathrm{St}((\pi^s_{n,G})^{\leftarrow}(\bigcup \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}F_m), \mathscr{U}_m))} \\ &\subseteq \quad \overline{\mathrm{St}(\bigcup \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}F_m, \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}\mathscr{U}_m)}. \end{aligned}$$

On the other hand, suppose that $(\pi^s_{n,G})^{\leftarrow}(\bigcup \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} F_m) \cap (\pi^s_{n,G})^{\leftarrow}(\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} U) \neq \emptyset$. Then also $\pi^s_{n,G}((\pi^s_{n,G})^{\leftarrow}(\bigcup \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} F_m)) \cap \pi^s_{n,G}((\pi^s_{n,G})^{\leftarrow}(\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} U)) \neq \emptyset$, so $\bigcup \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} F_m \cap \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} U \neq \emptyset$. It shows that the space $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} X$ is weakly strongly star-Menger.

($\Leftarrow$) Suppose that $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} X$ is a weakly strongly star-Menger space. Let $\{\mathscr{U}_m\}_{m\in\mathbb{N}}$ be a sequence of open covers of X^n and $[\mathbf{x}]_G \in \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} X$. Since $(\pi^s_{n,G})^{\leftarrow}([\mathbf{x}]_G)$ is finite, for every $m \in \mathbb{N}$ there exists a finite subfamily $\mathscr{U}_{m,[\mathbf{x}]_G}$ of $\mathscr{U}_m$ such that $(\pi^s_{n,G})^{\leftarrow}([\mathbf{x}]_G) \subset \cup \mathscr{U}_{m,[\mathbf{x}]_G}$ and $U \cap (\pi^s_{n,G})^{\leftarrow}([\mathbf{x}]_G) \neq \emptyset$ for every $U \in \mathscr{U}_{m,[\mathbf{x}]_G}$. Since the mapping $\pi^s_{n,G}$ is closed, due to Lemma 3.1, there is an open neighborhood $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} V_{m,[\mathbf{x}]_G}$ of $[\mathbf{x}]_G$ in $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} X$ such that $(\pi^s_{n,G})^{\leftarrow}(\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} V_{m,[\mathbf{x}]_G}) \subset \cup\{U : U \in \mathscr{U}_{m,[\mathbf{x}]_G}\}$. Since the mapping $\pi^s_{n,G}$ is open, we have that $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} V_{m,[\mathbf{x}]_G} \subseteq \cap\{\pi^s_{n,G}(U) : U \in \mathscr{U}_{m,[\mathbf{x}]_G}\}$. Put $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} \mathscr{V}_m = \{\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} V_{m,[\mathbf{x}]_G} : [\mathbf{x}]_G \in \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} X\}$ for any $m \in \mathbb{N}$. Clearly, $\{\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} \mathscr{V}_m\}_{m\in\mathbb{N}}$ is a sequence of open covers of $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} X$. Since $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} X$ is weakly strongly star-Menger there exists a sequence $\{\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} F_m\}_{m\in\mathbb{N}}$ of finite subsets of $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} X$ such that $\overline{\bigcup_{m\in\mathbb{N}} \mathrm{St}(\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} F_m, \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} \mathscr{V}_m)} = \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} X$. Put $F_m = (\pi^s_{n,G})^{\leftarrow}(\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} F_m)$. Since the mapping $\pi^s_{n,G}$ is finite-to-one, a sequence $\{F_m\}_{m\in\mathbb{N}}$ is a sequence of finite subsets of X^n. Now we show that $\overline{\bigcup_{m\in\mathbb{N}} \mathrm{St}(F_m, \mathscr{U}_m)} = X^n$.

Let $\mathbf{x} \in X^n$ and V be an arbitrary neighbourhood of $\mathbf{x}$. Since the mapping $\pi^s_{n,G}$ is open, $\pi^s_{n,G}(V) = \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} V$ is a neighbourhood of $\pi^s_{n,G}(\mathbf{x}) = [\mathbf{x}]_G$. Then there exist $m \in \mathbb{N}$ and $[\mathbf{y}]_G \in \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} X$ such that $[\mathbf{x}]_G \in \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} V \cap \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} V_{m,[\mathbf{y}]_G}$ with $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} V_{m,[\mathbf{y}]_G} \cap \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} F_m \neq \emptyset$. We can choose $U \in \mathscr{U}_{m,[\mathbf{y}]_G}$ such that $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} V_{m,[\mathbf{y}]_G} \subseteq \pi^s_{n,G}(U)$. Since $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} V_{m,[\mathbf{y}]_G} \cap \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} F_m \neq \emptyset$, we have that $U \cap F_m \neq \emptyset$. Therefore, $\mathbf{x} \in \overline{\bigcup_{m\in\mathbb{N}} \mathrm{St}(F_m, \mathscr{U}_m)}$. Thus shows that X^n is a weakly strongly star-Menger space. Theorem 3.12 is proved. $\square$

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