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Family of Occasionally Weakly Compatible Mappings with an Application in Dynamic Programming

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Abstract: In this paper, we investigate the existence of a unique common fixed point of families of occasionally weakly compatible mappings along with property$(E.A)$ satisfying a generalized (ψ,ϕ)-weak contraction condition involving cubic terms of distance function which generalize some known results. As an application, we discuss the existence and uniqueness of a common solution of certain functional equations arising in dynamic programming.

Keywords: (ψ,ϕ)-weak contraction, Occasionally weakly compatible mappings, property $(E.A)$, Functional equations, Dynamic programming.

1 Introduction and preliminaries

Banach contraction principle [6] is the basic result of fixed point theory which states that every contraction mapping T(say) defined on a complete metric space E(say) has a unique common fixed point. For the last ten decades, many researchers are trying to generalize and extend this basic result in various directions. In 1976, Jungck [21] used the notion of commuting mappings for the generalization of Banach contraction principle. In 1982, Sessa [33] relaxed the commutative condition of mapping to weak commutative mappings. Further, in 1986, Jungck [22] introduced the notion of compatible mappings to weakened the notion of commutativity/weak commutativity of mappings as follows:

Definition 1.1. *[22] Two self mappings S and T of a metric space (E,d) are said to be compatible if and only if*

$$\lim_{n\to\infty} d(STu_n, TSu_n) = 0,$$

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whenever $\{u_n\}$ *is a sequence in* E *such that* $\lim_{n\to\infty} Su_n = \lim_{n\to\infty} Tu_n = z$*, for some* $z \in E$.

In 1996, the concept of weakly compatible mappings was introduced by Jungck [23] which may be consider as the minimal commutativity of mappings.

Definition 1.2. *[23] Let* S *and* T *be two self mappings of a metric space* (E,d)*. Then* S *and* T *are said to be weakly compatible mappings if the mappings commute at their coincidence points.*

In 2002, Aamri and Moutawakil [1] introduced a generalization of noncompatible mappings in the form of property $(E.A)$.

Definition 1.3. *[1] Two self mappings* S *and* T *defined on a metric space* (E,d) *are said to satisfy property* $(E.A)$ *if there exists a sequence* $\{u_n\}$ *in* E *such that*

$$\lim_{n\to\infty} Su_n = \lim_{n\to\infty} Tu_n = z, \text{ for some } z \in E.$$

In 2008, Al-Thagafi and Shahzad[3] introduced notion of occasionally weakly compatible mapping to generalize the notion of weakly compatible mappings as follows.

Definition 1.4. *[3] The pair* (S,T) *is said to be occasionally weakly compatible, if there exists a coincidence point* $u \in E$ *such that* $Su = Tu$ *implies* $STu = TSu$.

Remark 1.1. *Weakly compatible mappings are occasionally weakly compatible mappings but converse need not true (see [4]).*

Remark 1.2. *Occasionally weakly compatible mappings and property* $(E.A)$ *are independent of each other (see [5]).*

In 1971, Ćirić[10] investigated a class of self mappings on a metric space (E,d) satisfying the following condition.

$$d(fu,gv) \leq k\max\{d(u,v),d(u,fu),d(v,fv),\frac{1}{2}[d(u,fv)+d(v,fu)]\}, \tag{1}$$

where $0 < k < 1$. In 1974, Ćirić [11] proved common fixed point theorem for a family of mappings satisfying the condition (1) as follows.

Theorem 1.1. *[11] Let* (E,d) *be a complete metric space and* $\{T_i\}_{i\in\Lambda}$ *be a family of self mappings defined on* E*. If there exists a fixed* $j \in \Lambda$ *such that for each* $i \in \Lambda$ *and all* $u,v \in E$

$$d(T_iu,T_jv) \leq \lambda \max\{d(u,v),d(u,T_iu),d(v,T_jv),\frac{1}{2}[d(u,T_jv)+d(v,T_iu)]\},$$

where $\lambda = \lambda(i) \in (0,1)$*, then all* T_i *have a unique common fixed point in* E.

In 2005, Singh and Jain [32] proved the following fixed point theorem for commuting self mappings.

Theorem 1.2. *[32] Let (E,d) be a complete metric space and A,B,P,Q,S and T be self mappings on E such that*

(H_1) *$P(E) \subset ST(E)$, $Q(E) \subset AB(E)$;*

(H_2) *$ST = TS$, $PB = BP$, $AB = BA$, $QT = TQ$;*

(H_3) *either AB or P is continuous;*

(H_4) *the pair (Q,ST) is weakly compatible and the pair (P,AB) is compatible;*

(H_5) *for all $u,v \in E$ and for some k , $0 < k < 1$,*

$$d(Pu,Qv) \leq k\max\{d(Pu,ABv),d(Qv,STv),d(ABu,STv),$$

$$\frac{1}{2}[d(Pu,STv)+d(Mv,ABu)]\}.$$

Then P,Q,S,T,A and B have a unique common fixed point.

In 2008, Ćirić *et al.* [12] proved common fixed point theorems for family of mappings satisfying generalized non-linear contraction condition of type (1) in metric spaces and generalized the result of Singh and Jain [32].

Another direction of generalization of Banach contraction principle concerns with the use of control function. In 1969, Boyd and Wong [8] introduced ϕ contraction of the form $d(Tu,Tv) \leq \phi(d(u,v))$, for all $u,v \in E$, where T is a self mapping on a complete metric space E and $\phi : [0,\infty) \to [0,\infty)$ is an upper semi continuous function from right such that $0 \leq \phi(t) < t$, for all $t > 0$. In 1997, Alber and Guerre- Delabriere [2] generalized ϕ contraction to $\phi-$weak contraction in Hilbert spaces, which was further extended and proved by Rhoades [31] in complete metric space.

A self mapping T on a complete metric space is said to be a $\phi-$ weak contraction if for each $u,v \in E$, there exists a continuous non-decreasing function $\phi : [0,\infty) \to [0,\infty)$ satisfying $\phi(t) > 0$, for all $t > 0$ and $\phi(t) = 0$ if and only if $t = 0$ such that

$$d(Tu,Tv) \leq d(u,v) - \phi(d(u,v)). \tag{2}$$

The function ϕ in the above inequality (2) is known as control function or altering distance function. The notion of control function was given by Khan *et al.* [26] : an altering distance is an increasing and continuous function $\phi : [0,\infty) \to [0,\infty)$ vanishing only at zero.

In 2009, Zhang and Song [35] gave the notion of generalized $\phi-$ weak contraction by generalizing the concept of $\phi-$weak contraction.

Theorem 1.3. *[35] Let (E,d) be a metric space and S and T be two self mappings defined on E such that for $u,v \in E$*

$$d(Su,Tv) \leq M(u,v) - \phi(M(u,v)), \tag{3}$$

where $M(u,v) = \max\{d(u,v), d(u,Su), d(v,Tv), \frac{d(u,Tv)+d(v,Su)}{2}\}$ *and* $\phi : [0,\infty) \to [0,\infty)$ *is a lower semi continuous function with* $\phi(t) > 0$*, for all* $t > 0$ *and* $\phi(0) = 0$*. Then, there exists a unique point* $u \in E$ *such that* $Su = u = Tu$*.*

In 2011, Razani and Yazdi [30] proved a common fixed point theorem for a family of compatible mappings satisfying generalized $\phi-$weak contraction condition of type (3).

In 2013, Murthy and Prasad [29] introduced a weak contraction involving cubic terms of distance function and proved the following fixed point theorem for a mapping.

Theorem 1.4. *[29] Let* T *be a self mapping on a complete metric space* E *satisfying:*

$$[1+pd(u,v)]d^2(Tu,Tv) \le p\max\Big\{\frac{1}{2}[d^2(u,Tu)d(v,Tv)+d(u,Tu)d^2(v,Tv)],$$

$$d(u,Tu)d(u,Tv)d(v,Tu), d(u,Tv)d(v,Tu)d(v,Tv)\Big\} + m(u,v) - \phi(m(u,v)),$$

where

$$m(u,v) = \max\Big\{d^2(u,v), d(u,Tu)d(v,Tv), d(u,Tv)d(v,Tu), \frac{1}{2}[d(u,Tu)d(u,Tv)+d(v,Tu)d(v,Tv)]\Big\},$$

where $p \ge 0$ *is a real number and* $\phi : [0,\infty) \to [0,\infty)$ *is a continuous function with* $\phi(t) = 0$ *if and only if* $t = 0$ *and* $\phi(t) > 0$ *for each* $t > 0$*. Then* T *has a unique fixed point in* E*.*

Theorem 1.4 was extended and generalized for a variety of commuting self mappings on metric space [15, 16, 17, 20, 25, 28]. In 2022, Kavita and Kumar [25] introduced a generalized (ψ,ϕ)-weak contraction involving cubic terms of metric function and generalized the Theorem 1.4. Motivated by Ćirić *et al.* [12], [13], we establish the existence and uniqueness of common fixed point for family of occasionally weakly compatible mappings satisfying a generalized (ψ,ϕ)-weak contraction involving cubic terms of metric function. These results generalize and extend the results of Ćirić [11], Ćirić *et al.*[12], Chugh and Kumar [9], Jain *et al.* [16, 18, 19, 20], Kang *et al.* [27], Murthy and Prasad [29], Razani and Yazdi [30] and Singh and Jain [32] and Zhang and Song [35]. Further, as an application of our result, we obtain a common solution to a certain system of functional equation arising in the dynamic programming.

2 Main Results

Let Ψ is a collection of all functions $\psi : [0,\infty)^4 \to [0,\infty)$ satisfying the following conditions:

(ψ_1) ψ is non decreasing and upper semi continuous in each coordinate variables,

(ψ_2) $\Delta(t) = \max\{\psi(t,t,0,0), \psi(0,0,0,t), \psi(0,0,t,0), \psi(t,t,t,t)\} \le t$, for each $t > 0$.

Let Φ be a collection of all the functions $\phi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(ϕ_1) ϕ is a continuous function,

(ϕ_2) $\phi(t) > 0$ for each $t > 0$ and $\phi(0) = 0$.

Throughout this section, $C(S,T)$ denotes the set of coincidences points of mappings S and T and let $A' = A_2A_4\cdots A_{2n}$ and $A'' = A_1A_3\cdots A_{2n-1}$, where A_i, $i = 1,2,\ldots,2n$ are as mentioned in the following theorems.

Theorem 2.1. *Let S, T and A_i, $i = 1\ldots 2n$, be self mappings of a metric space (E,d) satisfying the following conditions:*

(C_1) *for all $u,v \in E$, there exists functions $\psi \in \Psi$ and $\phi \in \Phi$, a real number $p > 0$ such that*

$$[1+pd(A'u,A''v)]d^2(Su,Tv) \le p\psi\Bigg(d^2(A'u,Su)d(A''v,Tv), d(A'u,Su)d^2(A''v,Tv),$$

$$d(A'u,Su)d(A'u,Tv)d(A''v,Su), d(A'u,Tv)d(A''v,Su)d(A''v,Tv)\Bigg)+$$

$$+m(A'u,A''v) - \phi(m(A_2\cdots A_{2n}u,A''v)),$$

where

$$m(A'u,A''v) = \max\Big\{d^2(A'u,A''v), d(A'u,Su)d(A''v,Tv), d(A'u,Tv)d(A''v,Su),$$
$$\frac{1}{2}[d(A'u,Su)d(A'u,Tv)+d(A''v,Su)d(A''v,Tv)]\Big\}.$$

(C_2) *Suppose that either*

(a) $T(E) \subset A'(E)$ and the pair (T,A'') satisfy the property $(E.A)$ and $A''(E)$ is a closed subset of E; or

(b) $S(E) \subset A''(E)$ and the pair (S,A') satisfy the property $(E.A)$ and $A'(E)$ is a closed subset of E

Then $C(S,A') \neq \emptyset$ and $C(T,A'') \neq \emptyset$.

Proof. *Suppose (a) holds.*
Since the pairs (T,A'') satisfy the property $(E.A)$, there exist a sequence $\{v_k\}$ in E such that

$$\lim_{k\to\infty} A''v_k = \lim_{k\to\infty} Tv_k = w, \text{ for some } w \in E.$$

Since $T(E) \subset A'(E)$ there exists a sequence $\{u_k\}$ such that $Tv_k = A'u_k$. Hence, $\lim_{k\to\infty} A'u_k = w$. We claim that $\lim_{k\to\infty} Su_k = w$. Taking $u = u_k$ and $v = v_k$ in (C_1), we have

$$[1+pd(A'u_k,A''v_k)]d^2(Su_k,Tv_k) \le p\psi\Bigg(d^2(A'u_k,Su_k)d(A''v_k,Tv_k),$$

$$d(A'u_k,Su_k)d^2(A''v_k,Tv_k), d(A'u_k,Su_k)d(A'u_k,Tv_k)d(A''v_k,Su_k),$$

$$d(A'u_k,Tv_k)d(A''v_k,Su_k)d(A''v_k,Tv_k)\Bigg) + m(A'u_k,A''v_k) - \phi(m(A_2\cdots A_{2n}u_k,A''v_k)),$$

where

$$m(A'u_k,A''v_k) = \max\Big\{d^2(A'u_k,A''v_k), d(A'u_k,Su_k)d(A''v_k,Tv_k),$$

$$d(A'u_k,Tv_k)d(A''v_k,Su_k), \frac{1}{2}[d(A'u_k,Su_k)d(A'u_k,Tv_k) + d(A''v_k,Su_k)d(A''v_k,Tv_k)]\Big\}.$$

Letting $k \to \infty$

$$[1+pd(w,w)]d^2(\lim_{k\to\infty} Su_k,w) \le p\psi\Bigg(d^2(w,\lim_{k\to\infty} Su_k)d(w,w), d(w,\lim_{k\to\infty} Su_k)d^2(w,w),$$

$$d(w,\lim_{k\to\infty} Su_k)d(w,w)d(w,\lim_{k\to\infty} Su_k), d(w,w)d(w,\lim_{k\to\infty} Su_k)d(w,w)\Bigg)$$

$$+m(w,w) - \phi(m(w,w)),$$

where

$$m(w,w) = \max\Big\{d^2(w,w), d(w,\lim_{k\to\infty} Su_k)d(w,w), d(w,w)d(w,\lim_{k\to\infty} Su_k),$$

$$\frac{1}{2}[d(w,\lim_{k\to\infty} Su_k)d(w,w) + d(w,\lim_{k\to\infty} Su_k)d(w,w)]\Big\} = 0.$$

Simplifying the above inequality, we have $d^2(\lim_{k\to\infty} Su_k,w) \le 0$, which holds only for $\lim_{k\to\infty} Su_k = w$.

Since $A''(E)$ is a closed subset of E, then there exists $z \in E$ such that $w = A''z$. Now, we prove that $Tz = w$, for this taking $u = u_k$ and $v = z$ in (C_1), we have

$$[1+pd(A'u_k,A''z)]d^2(Su_k,Tz) \le p\psi\Bigg(d^2(A'u_k,Su_k)d(A''z,Tz),$$

$$d(A'u_k,Su_k)d^2(A''z,Tz), d(A'u_k,Su_k)d(A'u_k,Tz)d(A''z,Su_k),$$

$$d(A'u_k,Tz)d(A''z,Su_k)d(A''z,Tz)\Bigg) + m(A'u_k,A''z) - \phi(m(A_2\cdots A_{2n}u_k,A''z)),$$

where

$$m(A'u_k, A''z) = \max\Big\{d^2(A'u_k, A''z), d(A'u_k, Su_k)d(A''z, Tz), d(A'u_k, Tz)d(A''z, Su_k),$$
$$\frac{1}{2}[d(A'u_k, Su_k)d(A'u_k, Tz) + d(A''z, Su_k)d(A''z, Tz)]\Big\}.$$

Letting $k \to \infty$

$$[1 + pd(w,w)]d^2(w,Tz) \le p\psi(0,0,0,0) + m(w,w) - \phi(m(w,w)),$$

where

$$m(w,w) = \max\Big\{d^2(w,w), d(w,w)d(w,Tz), d(w,Tz)d(w,w),$$
$$\frac{1}{2}[d(w,w)d(w,Tz) + d(w,w)d(w,Tz)]\Big\} = 0.$$

After simplification, we get $d^2(w,Tz) \le 0$*, which is true only for* $Tz = w$*. Hence,* $A''z = w = Tz$*, i.e.,* $C(T, A'') \neq \emptyset$.
Since $T(E) \subset A'(E)$*, there exists* $x \in E$ *such that* $w = Tz = A'x$*. Now, we claim that* $Sx = w$*. Substituting* $u = x$ *and* $v = z$ *in* (C_1)*, we get*

$$[1 + pd(w,w)]d^2(Sx,w) \le p\psi(0,0,0,0) + m(w,w) - \phi(m(w,w)),$$

where

$$m(w,w) = \max\Big\{d^2(w,w), d(w,Sx)d(w,w), d(w,w)d(w,Sx),$$
$$\frac{1}{2}[d(w,Sx)d(w,w) + d(w,Sx)d(w,w)]\Big\} = 0.$$

Simplifying the above inequality, we get $d^2(Sx,w) \le 0$*, which is possible only if* $Sx = w$*. Hence,* $A'x = w = Sx$*, i.e.,* $C(S, A') \neq \emptyset$.
Similarly, the assertion of the theorem are true under the assumption (b).

Now, we establish the existence of a unique common fixed point for even number of occasionally weakly compatible mappings.

Theorem 2.2. *Let* S, T*,* $A_i (i = 1, 2, \ldots, 2n)$ *be self mappings of a metric space* (E,d) *satisfying the conditions* (C_1)*,* (C_2) *and*

(C_3) $A_1(A_3 \cdots A_{2n-1}) = (A_3 \cdots A_{2n-1})A_1$,
$A_1A_3(A_5 \cdots A_{2n-1}) = (A_5 \cdots A_{2n-1})A_1A_3$,
...
$A_1A_3 \cdots A_{2n-3}(A_{2n-1}) = (A_{2n-1})A_1A_3 \cdots A_{2n-3}$;
$T(A_3 \cdots A_{2n-1}) = (A_3 \cdots A_{2n-1})T$,
$T(A_5 \cdots A_{2n-1}) = (A_5 \cdots A_{2n-1})T$,
...

$TA_{2n-1} = A_{2n-1}T$;
$A_2(A_4 \cdots A_{2n}) = (A_4 \cdots A_{2n})A_2$,
$A_2A_4(A_6 \cdots A_{2n}) = (A_6 \cdots A_{2n})A_2A_4$,
...
$A_2A_4 \cdots A_{2n-2}(A_{2n}) = (A_{2n})A_2A_4 \cdots A_{2n-2}$;
$S(A_4 \cdots A_{2n}) = (A_4 \cdots A_{2n})S$,
$S(A_6 \cdots A_{2n}) = (A_6 \cdots A_{2n})S$,
...
$SA_{2n} = A_{2n}S$.

If the pairs (T,A'') and (S,A') are occasionally weakly compatible, then mappings $A_1,A_2,\ldots,$ A_{2n-1}, A_{2n}, S and T have a unique common fixed point in E.

Proof. *By Theorem 2.1, $C(T,A'') \neq \emptyset$ and $C(S,A') \neq \emptyset$. Since the pairs (S,A') is occasionally weakly compatible mappings, there exists a point $z \in C(S,A')$ such that $A'z = Sz = x$(say) and $A'Sz = SA'z = x^*$(say), hence $A'x = A'Sz = x^* = SA'z = Sx$.*

Since the pair (T,A'') is occasionally weakly compatible mappings, there exists a point $w \in C(T,A'')$ such that $A''w = Tw = y$(say) and $A''Tw = TA''w = y^*$(say), hence $A''y = A''Tw = y^* = TA''w = Ty$. We claim that $x^* = y^*$. Taking $u = x$ and $v = y$ in (C_1), we get

$$\begin{aligned}[1+pd(A'x,A''y)]d^2(Sx,Ty) \leq p\psi\Big(& d^2(A'x,Sx)d(A''y,Ty), d(A'x,Sx)d^2(A''y,Ty), \\ & d(A'x,Sx)d(A'x,Ty)d(A''y,Sx), d(A'x,Ty)d(A''y,Sx)d(A''y,Ty)\Big) \\ & + m(A'x,A''y) - \phi(m(A'x,A''y)),\end{aligned}$$

where

$$\begin{aligned}m(A'x,A''y) = \max\Big\{& d^2(A'x,A''y), d(A'x,Sx)d(A''y,Ty), d(A'x,Ty)d(A''y,Sx), \\ & \frac{1}{2}[d(A'x,Sx)d(A'x,Ty) + d(A''y,Sx)d(A''y,Ty)]\Big\}.\end{aligned}$$

Simplifying the above inequality, we have

$$[1+pd(x^*,y^*)]d^2(x^*,y^*) \leq p\psi(0,0,0,0) + m(x^*,y^*) - \phi(m(x^*,y^*)),$$

where

$$m(x^*,y^*) = \max\Big\{d^2(x^*,y^*), 0, d(x^*,y^*)d(y^*,x^*), 0\Big\} = d^2(x^*,y^*).$$

We conclude that $pd^3(x^*,y^*) + \phi(d^2(x^*,y^*)) \leq 0$, which is true only for $x^* = y^*$, hence, $A'x = Sx = x^*$ and $A''y = Ty = x^*$. Next, we claim that $x^* = x$. For this, taking $u = z$, $v = y$

in (C_1)*, we get*

$$[1+pd(A'z,A''y)]d^2(Sz,Ty) \leq p\psi\Big(d^2(A'z,Sz)d(A''y,Ty), d(A'z,Sz)d^2(A''y,Ty), d(A'z,Sz)d(A'z,Ty)d(A''y,Sz), d(A'z,Ty)d(A''y,Sz)d(A''y,Ty)\Big) + m(A'z,A''y) - \phi(m(A'z,A''y)),$$

where

$$m(A'z,A''y) = \max\Big\{d^2(A'z,A''y), d(A'z,Sz)d(A''y,Ty), d(A'z,Ty)d(A''y,Sz), \frac{1}{2}[d(A'z,Sz)d(A'z,Ty)+d(A''y,Sz)d(A''y,Ty)]\Big\},$$

which implies that

$$[1+pd(x,x^*)]d^2(x,x^*) \leq p\psi(0,0,0,0) + m(x,x^*) - \phi(m(x,x^*)),$$

where

$$m(x,x^*) = \max\Big\{d^2(x,x^*), 0, d(x,x^*)d(x^*,x), 0\Big\} = d^2(x,x^*).$$

After simplification, we get $pd^3(x,x^*) + \phi(d^2(x,x^*)) \leq 0$*, which holds only for* $x = x^*$*. Hence,* $Sx = A'x = x$ *and* $A''y = Ty = x$*. Further, we claim that* $x = y$*. Taking* $u = x$*,* $v = w$

$$[1+pd(A'x,A''w)]d^2(Sx,Tw) \leq p\psi\Big(d^2(A'x,Sx)d(A''w,Tw), d(A'x,Sx)d^2(A''w,Tw), d(A'x,Sx)d(A'x,Tw)d(A''w,Sx), d(A'x,Tw)d(A''w,Sx)d(A''w,Tw)\Big) + m(A'x,A''w) - \phi(m(A'x,A''w)),$$

where

$$m(A'x,A''w) = \max\Big\{d^2(A'x,A''w), d(A'x,Sx)d(A''w,Tw), d(A'x,Tw)d(A''w,Sx), \frac{1}{2}[d(A'x,Sx)d(A'x,Tw)+d(A''w,Sx)d(A''w,Tw)]\Big\},$$

which implies that

$$[1+pd(x,y)]d^2(x,y) \leq p\psi(0,0,0,0) + m(x,y) - \phi(m(x,y)),$$

where

$$m(x,y) = \max\Big\{d^2(x,y), 0, d(x,y)d(y,x), 0\Big\} = d^2(x,y).$$

After simplification, we get $pd^3(x,y)+\phi(d^2(x,y)) \le 0$*, which is possible only for* $x=y$ *and hence,* $A'x = Sx = A''x = Tx = x$.

Taking $u = A_4 \cdots A_{2n}x$ *and* $v = x$ *in* (C_1)*, we have*

$$[1+pd(A'A_4\cdots A_{2n}x, A''x)]d^2(SA_4\cdots A_{2n}x, Tx) \le$$

$$p\psi\Bigg(d^2(A'A_4\cdots A_{2n}x, SA_4\cdots A_{2n}x)d(A''x,Tx), d(A'A_4\cdots A_{2n}x, SA_4\cdots A_{2n}x)d^2(A''x,Tx),$$
$$d(A'A_4\cdots A_{2n}x, SA_4\cdots A_{2n}x)d(A'A_4\cdots A_{2n}x, Tx)d(A''x, SA_4\cdots A_{2n}x),$$
$$d(A'A_4\cdots A_{2n}x, Tx)d(A''x, SA_4\cdots A_{2n}x)d(A''x,Tx)\Bigg)+$$
$$+m(A'A_4\cdots A_{2n}x, A''x) - \phi(m(A'A_4\cdots A_{2n}x, A''x)),$$

where

$$\begin{aligned} m(A'A_4\cdots A_{2n}x, A''x) = \max\Big\{ & d^2(A'A_4\cdots A_{2n}x, A''x), \\ & d(A'A_4\cdots A_{2n}x, SA_4\cdots A_{2n}x)d(A''x,Tx), \\ & d(A'A_4\cdots A_{2n}x, Tx)d(A''x, SA_4\cdots A_{2n}x), \\ & \frac{1}{2}[d(A'A_4\cdots A_{2n}x, SA_4\cdots A_{2n}x)d(A'A_4\cdots A_{2n}x, Tx) \\ & + d(A''x, SA_4\cdots A_{2n}x)d(A''x,Tx)]\Big\}. \end{aligned}$$

Applying condition (C_3) *and* $Tx = A''x = x$*, we have*

$$[1+pd(A_4\cdots A_{2n}x,x)]d^2(A_4\cdots A_{2n}x,x) \le p\psi\Bigg(d^2(A_4\cdots A_{2n}x, A_4\cdots A_{2n}x)d(x,x),$$
$$d(A_4\cdots A_{2n}x, A_4\cdots A_{2n}x)d^2(x,x), d(A_4\cdots A_{2n}x, A_4\cdots A_{2n}x)d(A_4\cdots A_{2n}x,x)d(x, A_4\cdots A_{2n}x),$$
$$d(A_4\cdots A_{2n}x,x)d(x,A_4\cdots A_{2n}x)d(x,x)\Bigg) + m(A_4\cdots A_{2n}x,x) - \phi(m(A_4\cdots A_{2n}x,x)),$$

where

$$\begin{aligned} m(A_4\cdots A_{2n}x,x) = \max\Big\{ & d^2(A_4\cdots A_{2n}x,x), \\ & d(A_4\cdots A_{2n}x, A_4\cdots A_{2n}x)d(x,x), \\ & d(A_4\cdots A_{2n}x,x)d(x,A_4\cdots A_{2n}x), \\ & \frac{1}{2}[d(A_4\cdots A_{2n}x, A_4\cdots A_{2n}x)d(A_4\cdots A_{2n}x,x) \\ & + d(x,A_4\cdots A_{2n}x)d(x,x)]\Big\}, \end{aligned}$$

which implies that

$$[1+pd(A_4\cdots A_{2n}x,x)]d^2(A_4\cdots A_{2n}x,x)\leq p\psi(0,0,0,0)+m(A_4\cdots A_{2n}x,x) - \phi(m(A_4\cdots A_{2n}x,x)),$$

where

$$m(A_4\cdots A_{2n}x,x)=\max\{d^2(A_4\cdots A_{2n}x,x),0,d^2(A_4\cdots A_{2n}x,x),0\}=d^2(A_4\cdots A_{2n}x,x),$$

which becomes $pd^3(A_4\cdots A_{2n}x,x)+\phi(d^2(A_4\cdots A_{2n}x,x))\leq 0$*, which holds for* $A_4\cdots A_{2n}x=x$*. Therefore,* $x=A'x=A_2A_4\cdots A_{2n}x=A_2x$*.*
Continuing in this manner, we get $Sx=A_2x=A_4x=...=A_{2n}x=x$*.*

Taking $u=x$ *and* $v=A_3\cdots A_{2n-1}x$ *in* (C_1)*, we have*

$$\begin{aligned}&[1+pd(A'x,A''A_3\cdots A_{2n-1}x)]d^2(Sx,TA_3\cdots A_{2n-1}x)\leq\\ &p\,\psi\Bigg(d^2(A'x,Sx)d(A''A_3\cdots A_{2n-1}x,TA_3\cdots A_{2n-1}x),\\ &d(A'x,Sx)d^2(A''A_3\cdots A_{2n-1}x,TA_3\cdots A_{2n-1}x),\\ &d(A'x,Sx)d(A'x,TA_3\cdots A_{2n-1}x)d(A''A_3\cdots A_{2n-1}x,Sx),\\ &d(A'x,TA_3\cdots A_{2n-1}x)d(A''A_3\cdots A_{2n-1}x,Sx)d(A''A_3\cdots A_{2n-1}x,TA_3\cdots A_{2n-1}x)\Bigg)+\\ &+m(A'x,A''A_3\cdots A_{2n-1}x)-\phi(m(A'x,A''A_3\cdots A_{2n-1}x)),\end{aligned}$$

where

$$\begin{aligned}&m(A'x,A''A_3\cdots A_{2n-1}x)=\max\Big\{d^2(A'x,A''A_3\cdots A_{2n-1}x),\\ &(A'x,Sx)d(A''A_3\cdots A_{2n-1}x,TA_3\cdots A_{2n-1}x),d(A'x,TA_3\cdots A_{2n-1}x)d(A''A_3\cdots A_{2n-1}x,Sx),\\ &\frac{1}{2}[d(A'x,Sx)d(A'x,TA_3\cdots A_{2n-1}x)+\\ &+d(A''A_3\cdots A_{2n-1}x,Sx)d(A''A_3\cdots A_{2n-1}x,TA_3\cdots A_{2n-1}x)]\Big\},\end{aligned}$$

Applying condition (C_3) *and* $A'x=Sx=x$*, we have*

$$\begin{aligned}&[1+pd(x,A_3\cdots A_{2n-1}x)]d^2(x,A_3\cdots A_{2n-1}x)\leq p\,\psi\Bigg(d^2(x,x)d(A_3\cdots A_{2n-1}x,A_3\cdots A_{2n-1}x),\\ &d(x,x)d^2(A_3\cdots A_{2n-1}x,A_3\cdots A_{2n-1}x),d(x,x)d(x,A_3\cdots A_{2n-1}x)d(A_3\cdots A_{2n-1}x,x),\\ &d(x,A_3\cdots A_{2n-1}x)d(A_3\cdots A_{2n-1}x,x)d(A_3\cdots A_{2n-1}x,A_3\cdots A_{2n-1}x)\Bigg)+\end{aligned}$$

$$+m(x, A_3 \cdots A_{2n-1}x) - \phi(m(x, A_3 \cdots A_{2n-1}x)),$$

where

$$m(x, A_3 \cdots A_{2n-1}x) = \max\Big\{d^2(x, A_3 \cdots A_{2n-1}x),$$

$$d(x,x)d(A_3 \cdots A_{2n-1}x, A_3 \cdots A_{2n-1}x), d(x, A_3 \cdots A_{2n-1}x)d(A_3 \cdots A_{2n-1}x, x),$$

$$\frac{1}{2}[d(x,x)d(x, A_3 \cdots A_{2n-1}x) + d(A_3 \cdots A_{2n-1}x, x)d(A_3 \cdots A_{2n-1}x, A_3 \cdots A_{2n-1}x)]\Big\},$$

After simplification, the above inequality becomes $pd^3(x, A_3 \cdots A_{2n-1}x) \leq 0$, *which is possible only if* $d(x, A_3 \cdots A_{2n-1}x) = 0$. *This implies that* $A_3 \cdots A_{2n-1}x = x$. *Therefore,* $Tx = A_1x = A_3x = ... = A_{2n-1}x = x$. *Hence,* $Sx = Tx = A_1x = A_2x = A_3x... = A_{2n-1}x = A_{2n}x = x$. *Uniqueness follows easily. Thus, x is a unique common fixed point of mappings S, T and $A_i (i = 1, 2, \ldots, 2n)$. This completes the proof.*

Now, we present slight generalized form of the above stated Theorems.

Theorem 2.3. *Let (E,d) be a metric space and let $\{S_\lambda\}_{\lambda \in \Lambda}$ and $A_i (i = 1, 2, \ldots, 2n)$ be two families of self mappings of E. Suppose there exists a fixed $\alpha \in \Lambda$ such that:*

(C_4) *for $\psi \in \Psi$, $\phi \in \Phi$, real number $p > 0$ and for all $u, v \in E$,*

$$[1 + pd(A'u, A''v)]d^2(S_\alpha u, S_\lambda v) \leq p\psi\Bigg(d^2(A'u, S_\alpha u)d(A''v, S_\lambda v),$$

$$d(A'u, S_\alpha u)d^2(A''v, S_\lambda v), d(A'u, S_\alpha u)d(A'u, S_\lambda v)d(A''v, S_\alpha u),$$

$$d(A'u, S_\lambda v)d(A''v, S_\alpha u)d(A''v, S_\lambda v)\Bigg) + m(A'u, A''v) - \phi(m(A'u, A''v)),$$

where

$$m(A'u, A''v) = \max\Big\{d^2(A'u, A''v), d(A'u, S_\alpha u)d(A''v, S_\lambda v), d(A'u, S_\lambda v)d(A''v, S_\alpha u),$$
$$\frac{1}{2}[d(A'u, S_\alpha u)d(A'u, S_\lambda v) + d(A''v, S_\alpha u)d(A''v, S_\lambda v)]\Big\}.$$

(C_5) $A_1(A_3 \cdots A_{2n-1}) = (A_3 \cdots A_{2n-1})A_1$,
$A_1A_3(A_5 \cdots A_{2n-1}) = (A_5 \cdots A_{2n-1})A_1A_3$,
...
$A_1A_3 \cdots A_{2n-3}(A_{2n-1}) = (A_{2n-1})A_1A_3 \cdots A_{2n-3}$;
$S_\lambda(A_3 \cdots A_{2n-1}) = (A_3 \cdots A_{2n-1})S_\lambda$,
$S_\lambda(A_5 \cdots A_{2n-1}) = (A_5 \cdots A_{2n-1})S_\lambda$,
...
$S_\lambda A_{2n-1} = A_{2n-1}S_\lambda$;
$A_2(A_4 \cdots A_{2n}) = (A_4 \cdots A_{2n})A_2$,
$A_2A_4(A_6 \cdots A_{2n}) = (A_6 \cdots A_{2n})A_2A_4$,

...
$A_2A_4\cdots A_{2n-2}(A_{2n}) = (A_{2n})A_2A_4\cdots A_{2n-2}$;
$S_\alpha(A_4\cdots A_{2n}) = (A_4\cdots A_{2n})S_\alpha$,
$S_\alpha(A_6\cdots A_{2n}) = (A_6\cdots A_{2n})S_\alpha$,
...
$S_\alpha A_{2n} = A_{2n}S_\alpha$;

(C_6) *Suppose that either*

(a) *$S_\lambda(E) \subset A'(E)$ and the pair (S_λ, A'') satisfy the property $(E.A)$ and $A''(E)$ is a closed subset of E; or*

(b) *$S_\alpha(E) \subset A''(E)$ and the pair (S_α, A') satisfy the property $(E.A)$ and $A'(E)$ is a closed subset of E*

Then the pairs (S_α, A') and (S_λ, A'') have a coincidence point each. Moreover, all S_λ and A_i have a unique common fixed point in E, if the pairs (S_λ, A'') and (S_α, A') are occasionally weakly compatible mappings.

Proof. *Let $S_{\lambda_0} \in \{S_\lambda\}_{\lambda\in\Lambda}$ be fixed. By taking $S = S_\alpha$, $T = S_{\lambda_0}$ and applying Theorems 2.1 and 2.2, it follows that there exists some $z \in E$ such that*

$$S_\alpha z = S_{\lambda_0} z = A_1 z = A_2 z = A_3 z = ... = A_{2n-1} z = A_{2n} z = z.$$

Let $\lambda \in \Lambda$ be arbitrary. Then, by taking $u = v = z$ in (C_4), we get

$$\begin{aligned}[1+pd(A'z,A''z)]d^2(S_\alpha z,S_\lambda z) \le p\psi\Big(& d^2(A'z,S_\alpha z)d(A''z,S_\lambda z), d(A'z,S_\alpha z)d^2(A''z,S_\lambda z),\\ & d(A'z,S_\alpha z)d(A'z,S_\lambda z)d(A''z,S_\alpha z),\\ & d(A'z,S_\lambda z)d(A''z,S_\alpha z)d(A''z,S_\lambda z)\Big)\\ & + m(A'z,A''z) - \phi(m(A'z,A''z)),\end{aligned}$$

where

$$\begin{aligned}m(A'z,A''z) = \max\Big\{& d^2(A'z,A''z), d(A'z,S_\alpha z)d(A''z,S_\lambda z), d(A'z,S_\lambda z)d(A''z,S_\alpha z),\\ & \frac{1}{2}[d(A'z,S_\alpha z)d(A'z,S_\lambda z) + d(A''z,S_\alpha z)d(A''z,S_\lambda z)]\Big\}.\end{aligned}$$

Simplifying the above inequality becomes $d^2(S_\lambda z, z) \le 0$, which is true only for $S_\lambda z = z$. Since λ was arbitrary, therefore $S_\lambda z = z$, for each $\lambda \in \Lambda$. Uniqueness follows easily. Thus, all S_λ and A_i have a unique common fixed point in E.

Remark 2.1. *Theorems 2.2 and 2.3 generalize the result of Ćirić et al. [11, 12] and Razani and Yazadi [30] for family of mappings.*

Remark 2.2. *Taking $n = 2$ in Theorem 2.2, we obtain a generalized version of Theorem 1.2 for six mappings.*

Remark 2.3. *Taking $n = 1$ in Theorem 2.2, we get following theorem, which extend and generalize the results of Ćirić[10], Chugh and Kumar [9], Jain et al. [16, 18, 19, 20], Jungck [21], Kang et al. [27], Murthy and Prasad [29] and Zhang and Song [35] in various aspects.*

Theorem 2.4. *Let (E,d) be a metric space and S,T,A_1 and A_2 be four self mappings of E satisfying the following conditions*

(C_1*) *for all $u,v \in E$, there exists functions $\phi \in \Phi$ and $\psi \in \Psi$ with a positive real number p such that*

$$[1+pd(A_2u,A_1v)]d^2(Su,Tv) \leq p\psi\bigg(d^2(A_2u,Su)d(A_1v,Tv),$$

$$d(A_2u,Su)d^2(A_1v,Tv), d(A_2u,Su)d(A_2u,Tv)d(A_1v,Su),$$

$$d(A_2u,Tv)d(A_1v,Su)d(A_1v,Tv)\bigg) + m(A_2u,A_1v) - \phi(m(A_2u,A_1v)),$$

where

$$m(A_2u,A_1v) = \max\Big\{d^2(A_2u,A_1v), d(A_2u,Su)d(A_1v,Tv), d(A_2u,Tv)d(A_1v,Su),$$
$$\frac{1}{2}[d(A_2u,Su)d(A_2u,Tv) + d(A_1v,Su)d(A_1v,Tv)]\Big\}.$$

(C_2*) *assume that either of the following holds*

(a) $T(E) \subset A_2(E)$ and the pair (T,A_1) satisfy the property $(E.A)$ and $A_1(E)$ is a closed subset of E;

(b) $S(E) \subset A_1(E)$ and the pair (S,A_2) satisfy the property $(E.A)$ and $A_2(E)$ is a closed subset of E.

Then the pairs (S,A_2) and (T,A_1) have a coincidence points each. Moreover, if the pairs (S,A_2) and (T,A_1) are occasionally weakly compatible, then S,T,A_1 and A_2 have a unique common fixed point in E.

Example 2.1. *Let $E = [0,10]$ and d be a usual metric. Let $A_1,A_2,S,T : E \to E$ be four mappings defined by $A_2u = 0, u = 0, A_2u = 8, u \in (0,\frac{5}{2}], A_2u = u - \frac{5}{2}, u \in (\frac{5}{2},10]$; $Su = 0, u \in (\frac{5}{2},10] \cup \{0\}, Su = 4, u \in (0,\frac{5}{2}]$; $A_1u = 0, u = 0, A_1u = 4, u \in (0,10]$; $Tu = 0, u = 0, Tu = 2, u \in (0,10]$. Let $p > 0$ be a real number and $\phi : [0,\infty) \to [0,\infty)$ be a function defined by $\phi(t) = \frac{3}{2}t$, for $t \geq 0$ and $\psi : [0,\infty)^4 \to [0,\infty)$ be a function defined by $\psi(t_1,t_2,t_3,t_4) = \max\{t_1,t_2,t_3,t_4\}$, $t_i \geq 0, i = 1,2,3,4$. Consider a sequence $\{u_n\} = \{2.5 + \frac{1}{n}\}$ in E. Then*

$\lim_{n\to\infty} A_2u_n = 0 = \lim_{n\to\infty} Su_n$, *but* $\lim_{n\to\infty} d(A_2Su_n, SA_2u_n) \neq 0$. *Hence, the pair* (A_2, S) *is not compatible but satisfy the property* $(E.A)$. *Also,* $S \subset A_2(E)$, $A_2(E)$ *is closed and for the sequence* $\{u_n\} = \{0\}$, *the pairs* (A_2, S) *and* (A_1, T) *are occasionally weakly compatible. Hence, all the conditions of the Theorem 2.2 are satisfied and* 0 *is the unique common fixed point of* A_2, A_1, S *and* T.

Corollary 2.1. *Let* (E, d) *be a metric space and* S *and* A *be two self mappings of* E *satisfying the following conditions*

(C_3*) *for all* $u, v \in E$, *there exists functions* $\phi \in \Phi$ *and* $\psi \in \Psi$ *with a positive real number* p *such that*

$$[1 + pd(Au,Av)]d^2(Su,Sv) \leq p\psi\Big(d^2(Au,Su)d(Av,Sv), d(Au,Su)d^2(Av,Sv), d(Au,Su)d(Au,Sv)d(Av,Su), d(Au,Sv)d(Av,Su)d(Av,Sv)\Big) + m(Au,Av) - \phi(m(Au,Av)),$$

where

$$m(Au,Av) = \max\Big\{d^2(Au,Av), d(Au,Su)d(Av,Sv), d(Au,Sv)d(Av,Su), \frac{1}{2}[d(Au,Su)d(Au,Sv) + d(Av,Su)d(Av,Sv)]\Big\},$$

(C_4*) $S(E) \subset A(E)$ *and* $A(E)$ *is a closed subset of* E,

(C_5*) *the pair* (S, A) *satisfy the property* $(E.A)$.

Then S *and* A *have a coincidence point. Moreover, if the pair* (S, A) *is occasionally weakly compatible, then* S *and* A *have a unique common fixed point in* E.
Proof. *Proof follows easily by taking* $S = T$ *and* $A_1 = A_2 = A$ *in Theorem 2.4.*

3 Application

Throughout this section, we assume that U and V are Banach spaces, $\hat{S} \subseteq U$ and $D \subseteq V$ are state and decision spaces respectively. Let $\mathbb{R}$ denote the field of real numbers and $B(\hat{S})$ denotes the set of all bounded real valued functions on S.

Bellman and Lee [7] presented the basic form of functional equation of dynamic programming as follows:

$$h(u) = opt_v\, G(u, v, h(\tau(u, v))),$$

where u and v are the state and decision vectors respectively, τ is the transformation of the process and $h(u)$ is the optimal return with initial state u and opt denotes max or min.

As an application of Theorem 2.4, we investigate the existence and uniqueness of a common solution of the following functional equations arising in dynamic programming.

$$h_i(u) = \sup_{v\in D} G_i(u,v,h_i(\tau(u,v))), u \in \hat{S}, \tag{4}$$

$$k_i(u) = \sup_{v\in D} F_i(u,v,k_i(\tau(u,v))), u \in \hat{S}, \tag{5}$$

where $\tau : \hat{S} \times D \to S$ and $G_i, F_i : \hat{S} \times D \times \mathbb{R} \to \mathbb{R}, i = 1,2$.

Define P_i and Q_i as follows

$$\begin{aligned} P_i f(u) &= \sup_{v\in D} F_i(u,v,f(\tau(u,v))), u \in \hat{S}, \\ Q_i g(u) &= \sup_{v\in D} G_i(u,v,g(\tau(u,v))), u \in \hat{S}, \end{aligned} \tag{6}$$

for all $u \in \hat{S}$; $f,g \in B(\hat{S})$, $i = 1,2$.

Theorem 3.1. *Suppose that the following conditions hold:*

(D_1) *G_i and F_i are bounded for $i = 1,2$.*

(D_2) *Assume that either (a) - (c) or (d) - (f) holds*

- *(a) for any $f \in B(\hat{S})$, there exists $g \in B(\hat{S})$ such that $Q_1 f(u) = P_2 g(u)$, $u \in \hat{S}$;*
- *(b) there exists $\{f_n\} \subset B(\hat{S})$ such that $\lim_{n\to\infty} P_1 f_n(u) = f(u) = \lim_{n\to\infty} Q_1 f_n(u)$, for some $f \in B(\hat{S})$;*
- *(c) for sequence $\{f_n\} \subset B(\hat{S})$ and $f \in B(\hat{S})$ with $\lim_{n\to\infty} P_1 f_n(u) = f(u)$, there exists $f^* \in B(\hat{S})$ such that $f(u) = P_1 f^*(u)$, for some $u \in \hat{S}$;*
- *(d) for any $k \in B(\hat{S})$, there exists $h \in B(\hat{S})$ such that $Q_2 k(u) = P_1 h(u)$, $u \in \hat{S}$;*
- *(e) there exists $\{g_n\} \subset B(\hat{S})$ such that $\lim_{n\to\infty} P_2 g_n(u) = g(u) = \lim_{n\to\infty} Q_2 g_n(u)$, for some $g \in B(\hat{S})$;*
- *(f) for sequence $\{g_n\} \subset B(\hat{S})$ and $g \in B(\hat{S})$ with $\lim_{n\to\infty} P_2 g_n(u) = g(u)$, there exists $g^* \in B(\hat{S})$ such that $g(u) = P_2 g^*(u)$, for some $u \in \hat{S}$.*

(D_3) *There exist $f,g \in B(\hat{S})$, $P_1 f = Q_1 f$ implies that $Q_1 P_1 f = P_1 Q_1 f$ and $P_2 g = Q_2 g$ implies that $P_2 Q_2 g = Q_2 P_2 g$.*

(D_4) *For all $(u,v) \in \hat{S} \times D$, $f,g \in B(\hat{S}), t \in \hat{S}$ such that*

$$\big|G_1(u,v,f(t)) - G_2(u,v,g(t))\big| \le M^{-1}\bigg(p\,\psi\Big(d^2(P_1 f, Q_1 f)d(P_2 g, Q_2 g),$$

$$d(P_1 f, Q_1 f)d^2(P_2 g, Q_2 g), d(P_1 f, Q_1 f)d(P_1 f, Q_2 g)d(P_2 g, Q_1 f),$$

$$d(P_1f,Q_2g)d(P_2g,Q_1f)d(P_2g,Q_2g)\Big)+m(P_1f,P_2g)-\phi(m(P_1f,P_2g))\Big),$$

where

$$m(P_1f,P_2g)=\max\big\{d^2(P_1f,P_2g),d(P_1f,Q_1f)d(P_2g,Q_2g),$$
$$d(P_1f,Q_2g)d(P_2g,Q_2g),\frac{1}{2}[d(P_1f,Q_1f)d(P_1f,Q_2g)]+d(P_2g,Q_1f)d(P_2g,Q_2g)\big\},$$

$M=\big[1+p\sup_{u\in\hat{S}}|P_1f(u)-P_2g(u)|\big]\sup_{u\in\hat{S}}|Q_1f(u)-Q_2g(u)|, Q_1f\neq Q_2g$, *$\phi\in\Phi$, $\psi\in\Psi$, p is a positive real number and the mappings P_1,P_2,Q_1 and Q_2 are defined as in (6).*

Then the system of functional equations given by (4) and (5) have a unique common solution in $B(\hat{S})$.

Proof. *Let $d(h,k)=\sup_{u\in\hat{S}}|h(u)-k(u)|$, for any $h,k\in B(\hat{S})$. Obviously, $(B(\hat{S}),d)$ is a complete metric space. For $\eta>0$, $u\in\hat{S}$ and $g_1,g_2\in B(\hat{S})$, there exists $v_1,v_2\in D$ such that*

$$Q_ig_i(u)<G_i(u,v_i,g_i(u_i))+\eta, \tag{7}$$

where $u_i=\tau(u,v_i), i=1,2$. Also, we have

$$Q_1g_1(u)\geq G_1(u,v_2,g_1(u_2)), \tag{8}$$
$$Q_2g_2(u)\geq G_2(u,v_1,g_2(u_1)). \tag{9}$$

From (7),(9) and (D_4), we have

$$\begin{aligned}Q_1g_1(u)-Q_2g_2(u)<&G_1(u,v_1,g_1(u_1))-G_2(u,v_1,g_2(u_1))+\eta\\ \leq& M^{-1}\bigg(p\,\psi\Big(d^2(P_1g_1,Q_1g_1)d(P_2g_2,Q_2g_2),\\ &d(P_1g_1,Q_1g_1)d^2(P_2g_2,Q_2g_2),\\ &d(P_1g_1,Q_1g_1)d(P_1g_1,Q_2g_2)d(P_2g_2,Q_1g_1),\\ &d(P_1g_1,Q_2g_2)d(P_2g_2,Q_1g_1)d(P_2g_2,Q_2g_2)\Big)\\ &+m(P_1g_1,P_2g_2)-\phi(m(P_1g_1,P_2g_2))\bigg)+\eta,\end{aligned} \tag{10}$$

From (7), (8) and (D_4), we have

$$\begin{aligned}Q_1g_1(u)-Q_2g_2(u)>&G_1(u,v_2,g_1(u_2))-G_2(u,v_2,g_2(u_2))-\eta\\ \geq& -M^{-1}\bigg(p\,\psi\Big(d^2(P_1g_1,Q_1g_1)d(P_2g_2,Q_2g_2),\\ &d(P_1g_1,Q_1g_1)d^2(P_2g_2,Q_2g_2),\\ &d(P_1g_1,Q_1g_1)d(P_1g_1,Q_2g_2)d(P_2g_2,Q_1g_1),\\ &d(P_1g_1,Q_2g_2)d(P_2g_2,Q_1g_1)d(P_2g_2,Q_2g_2)\Big)\\ &+m(P_1g_1,P_2g_2)-\phi(m(P_1g_1,P_2g_2))\bigg)-\eta,\end{aligned} \tag{11}$$

From (10) and (11), we obtain

$$
\begin{aligned}
|Q_1g_1(u)-Q_2g_2(u)| \leq M^{-1}\bigg(p\,\psi\Big(& d^2(P_1g_1,Q_1g_1)d(P_2g_2,Q_2g_2), \\
& d(P_1g_1,Q_1g_1)d^2(P_2g_2,Q_2g_2), \\
& d(P_1g_1,Q_1g_1)d(P_1g_1,Q_2g_2)d(P_2g_2,Q_1g_1), \\
& d(P_1g_1,Q_2g_2)d(P_2g_2,Q_1g_1)d(P_2g_2,Q_2g_2)\Big) \\
& +m(P_1g_1,P_2g_2)-\phi(m(P_1g_1,P_2g_2))\bigg)+\eta,
\end{aligned} \tag{12}
$$

As $\eta > 0$ is arbitrary and (12) is true for all $u \in \hat{S}$, taking supremum, we get

$$
\begin{aligned}
[1+pd(P_1g_1,P_2g_2)]d^2(Q_1g_1,Q_2g_2) \leq p\,\psi\Big(& d^2(P_1g_1,Q_1g_1)d(P_2g_2,Q_2g_2), \\
& d(P_1g_1,Q_1g_1)d^2(P_2g_2,Q_2g_2), \\
& d(P_1g_1,Q_1g_1)d(P_1g_1,Q_2g_2)d(P_2g_2,Q_1g_1), \\
& d(P_1g_1,Q_2g_2)d(P_2g_2,Q_1g_1)d(P_2g_2,Q_2g_2)\Big) \\
& +m(P_1g_1,P_2g_2)-\phi(m(P_1g_1,P_2g_2)).
\end{aligned} \tag{13}
$$

Therefore, Theorem 2.4 applies, where P_1,P_2,Q_1,Q_2 correspond to the mappings A_2,A_1,S,T respectively. So, P_1,P_2,Q_1 and Q_2 have a unique common fixed point $h^* \in B(\hat{S})$, i.e., $^*(u)$ is a unique common solution of the system of functional equations (4) and (5).

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